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EXPONENTIAL DECAY FOR A NONLINEAR VISCOELASTIC EQUATION WITH SINGULAR KERNELS*

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Abstract The nonlinear viscoelastic wave equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s + |u|^p u = 0,$$

in a bounded domain with initial conditions and Dirichlet boundary conditions is considered. We prove that, for a class of kernels g which is singular at zero, the exponential decay rate of the solution energy. The result is obtained by introducing an appropriate Lyapounov functional and using energy method similar to the work of Tatar in 2009. This work improves earlier results.

Key words viscoelastic wave equation; kernel; exponential decay; memory term; singular kernel

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1 Introduction

In this paper, we consider the following initial boundary value problem

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s + |u|^p u = 0 \text{ in } \Omega \times (0,\infty), \tag{1.1}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, t \geq 0, \tag{1.3}$$

where $\Omega \subset R^N$, $N \geq 1$, is a bounded domain with a smooth boundary $\partial\Omega$. Here ρ, $p > 0$ and the kernel $g(t)$ is of the form

$$g(t) = \frac{t^{-\alpha}\mathrm{e}^{-\beta t}}{\Gamma(1-\alpha)} \tag{1.4}$$

with $\beta > 0$ and $0 < \alpha < 1$. This type of equations usually appears as a model in nonlinear viscoelasticity when the material density varies according to the velocity. The memory

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term, represented by the convolution term in equation (1.1), expresses the mechanical response influenced by the history of the materials.

In [3], Cavalcanti et al. studied the following problem

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s - \gamma \Delta u_t = 0 \tag{1.5}$$

with the same initial and boundary conditions as (1.2)–(1.3). Under the assumption that $\gamma \geq 0$, $0 < \rho \leq \frac{2}{N-2}$ if $N \geq 3$ or $\rho > 0$ if $N = 1$, 2 and the function $g : R^+ \to R^+$ is a nonincreasing function, they proved a global existence result of weak solutions for $\gamma \geq 0$ and a uniform decay result for $\gamma > 0$. Precisely, they showed that the solution goes to zero in an exponential rate for $\gamma > 0$ and g is a positive bounded C^1-function satisfying

$$1 - \int_0^\infty g(s)\mathrm{d}s > 0 \quad \text{and} \quad -\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t) \tag{1.6}$$

for all $t \geq 0$ with some positive constants ξ_1 and ξ_2. Later, this result was extended by Messaoudi and Tatar [12] to the case $\gamma = 0$, where the exponential decay result was obtained without any mechanical dissipation other than the weak one produced by the viscoelastic term. Recently, Messaoudi and Tatar [11] studied problem (1.5) for the case of $\gamma = 0$, they improved the result in [3] by showing that the solution goes to zero with an exponential or polynomial rate, depending on the decay rate of the relaxation function g.

Assumptions (1.6), on g, are frequently encountered in the linear case ($\rho = 0$), see [1, 2, 4–6, 8–10, 13, 14, 17]. Lately, these conditions were weakened to enlarge the class of relaxation functions for which we have exponential decay. For example, Berrimi and Messaoudi [2] relaxed condition (1.6) to $g'(t) \leq -\xi g(t)$ for some positive constant ξ and $t > 0$. They established an exponential decay for the solution energy. Tatar [15] proved that the exponential decay for kernels with small L^1-norm under the conditions $g'(t) \leq 0$ and $\mathrm{e}^{\alpha t} g(t) \in L^1(0, \infty)$ for some large $\alpha > 0$. Under the assumption that the kernel is not necessarily nonincreasing, Medjden and Tatar [10] established an exponential decay of solutions under the assumptions $0 \leq (g'(t) + \xi g(t))\,\mathrm{e}^{\alpha t} \in L^1(0, \infty)$, for some $\alpha, \xi > 0$. Later on, instead of (1.6) Furati and Tatar [7] required the functions $\mathrm{e}^{\alpha t} g(t)$ and $\mathrm{e}^{\alpha t} g'(t)$ to have sufficiently small L^1-norm on $(0, \infty)$ for some $\alpha > 0$ and established an exponential decay of solutions. In particular, they do not impose a rate of decreasingness for g. Recently, Tatar [16] investigated the asymptotic behavior for problem (1.5) with $\rho = \gamma = 0$ and a memory term involving a kernel which has the feature to be singular at zero. The exponential decay result was obtained without setting any smallness condition on the L^1-norm of the kernel and without imposing conditions on the second and third order derivatives of the kernel.

Motivated by previous works, in this paper, we intend to study the exponential decay result of problem (1.1)–(1.3) for a class of kernels g which is singular at zero, while most of the existing works deal with functions defined at zero. Therefore, our result treats a larger class of relaxation functions and improves some earlier results concerning the exponential decay.

The content of this paper is organized as follows. In Section 2, we give some lemmas and assumptions which will be used later, and we present the local existence result Theorem 2.3. In Section 3, we establish the exponential decay result.