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APPROXIMATION OF A CAUCHY-JENSEN ADDITIVE FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN NORMED SPACES*

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Abstract Using the fixed point and direct methods, we prove the Hyers-Ulam stability of the following Cauchy-Jensen additive functional equation

$$2f\left(\frac{\sum\limits_{i=1}^{p} x_i + \sum\limits_{j=1}^{q} y_j + 2\sum\limits_{k=1}^{d} z_k}{2}\right) = \sum_{i=1}^{p} f(x_i) + \sum_{j=1}^{q} f(y_j) + 2\sum_{k=1}^{d} f(z_k),$$

where p, q, d are integers greater than 1, in non-Archimedean normed spaces.

Key words Hyers-Ulam stability; Cauchy-Jensen additive functional equation; fixed point; non-Archimedean normed spaces

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1 Introduction and Preliminaries

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?". If the problem accepts a solution, we say that the equation is stable. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940. In the next year, Hyers [2] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [3] proved a generalization of Hyers' theorem for additive mappings. Furthermore, in 1994, a generalization of Rassias' theorem was obtained by Găvruta [4] by replacing the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$.

In 1983, a Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [5] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. In 1984, Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group and, in 2002, Czerwik [7] proved the Hyers-Ulam stability of

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the quadratic functional equation. The reader is referred to [8–21] and references therein for detailed information on stability of functional equations.

In 1897, Hensel [22] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [24–28]).

Definition 1.1 By a non-Archimedean field we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|:\mathbb{K}\to[0,\infty)$ such that for all $r,s\in\mathbb{K}$, the following conditions hold:

(1) $|r|=0$ if and only if $r=0$;

(2) $|rs|=|r||s|$;

(3) $|r+s|\le\max\{|r|,|s|\}$.

Definition 1.2 Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\|:X\to R$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(1) $\|x\|=0$ if and only if $x=0$;

(2) $\|rx\|=|r|\|x\|\quad(r\in\mathbb{K},\ x\in X)$;

(3) The strong triangle inequality (ultrametric); namely, $\|x+y\|\le\max\{\|x\|,\|y\|\}$ for all $x,y\in X$. Then $(X,\|\cdot\|)$ is called a non-Archimedean space.

Due to the fact that $\|x_n-x_m\|\le\max\{\|x_{j+1}-x_j\|:m\le j\le n-1\}$ where $n>m$.

Definition 1.3 A sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1}-x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

Definition 1.4 Let X be a set. A function $d:X\times X\to[0,\infty]$ is called a generalized metric on X if d satisfies

(1) $d(x,y)=0$ if and only if $x=y$;

(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;

(3) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.5 Let (X,d) be a complete generalized metric space and $J:X\to X$ be a strictly contractive mapping with Lipschitz constant $\alpha<1$. Then for each given element $x\in X$, either $d(J^nx,J^{n+1}x)=\infty$ for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^nx,J^{n+1}x)<\infty,\ \forall n\ge n_0$;

(2) the sequence $\{J^nx\}$ converges to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in the set $Y=\{y\in X\mid d(J^{n_0}x,y)<\infty\}$;

(4) $d(y,y^*)\le\frac{1}{1-\alpha}d(y,Jy)$ for all $y\in Y$.

In 1996, Hyers, Isac and Rassias [28] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations were extensively investigated by a number of authors (see [29–32]).

This paper is organized as follows: In Section 2, using the fixed point method, we prove