Available online at www.sciencedirect.com

ScienceDirect

Acta Mathematica Scientia 2015,**35B**(2):383–398

Acta Mathematica Scientia 数学物理学报

http://actams.wipm.ac.cn

SEGREGATED VECTOR SOLUTIONS FOR NONLINEAR SCHRÖDINGER SYSTEMS IN $\mathbb{R}^2$*

Chunhua WANG (王春花)† *Dingyi XIE* (谢定一) *Liping ZHAN* (占丽萍)
Lipan ZHANG (张李攀) *Liangpei ZHAO* (赵良珮)
School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, China
E-mail: *chunhuawang@mail.ccnu.edu.cn*

Abstract We study the following nonlinear Schrödinger system

$$\begin{cases} -\Delta u + P(|x|)u = \mu u^3 + \beta v^2 u, & x \in \mathbb{R}^2, \\ -\Delta v + Q(|x|)v = \nu v^3 + \beta u^2 v, & x \in \mathbb{R}^2, \end{cases}$$

where $P(r)$ and $Q(r)$ are positive radial functions, $\mu > 0$, $\nu > 0$, and $\beta \in \mathbb{R}$ is a coupling constant. This type of system arises, particularly, in models in Bose-Einstein condensates theory. Applying a finite reduction method, we construct an unbounded sequence of non-radial positive vector solutions of segregated type when β is in some suitable interval, which gives an answer to an interesting problem raised by Peng and Wang in Remark 4.1 (Arch. Ration. Mech. Anal., 208(2013), 305-339).

Key words Segregated vector solutions; nonlinear Schrödinger systems

2010 MR Subject Classification 35J10; 35J15; 35J50

1 Introduction and Main Result

In this article, we are concerned with the following nonlinear Schrödinger system

$$\begin{cases} -\Delta u + P(|x|)u = \mu u^3 + \beta v^2 u, & x \in \mathbb{R}^2, \\ -\Delta v + Q(|x|)v = \nu v^3 + \beta u^2 v, & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where we suppose that $P(x)$ and $Q(x)$ are continuous positive radial functions, $\mu > 0$, $\nu > 0$, $\beta \in \mathbb{R}$ is a coupling constant.

These kinds of systems arise when one considers standing wave solutions of time-dependent N-coupled Schrödinger systems with $N = 2$ of the following form

$$\begin{cases} -\mathrm{i}\dfrac{\partial}{\partial t}\phi_j = \triangle\phi_j - V_j(x)\phi_j + \mu_j|\phi_j|^2\phi_j + \phi_j \displaystyle\sum_{l=1,k\neq j} \beta_{jk}|\phi_l|^2, & x \in \mathbb{R}^3, \\ -\phi_j = \phi_j(x,t) \in \mathbb{C}, t > 0, & j = 1, 2, \cdots, N, \end{cases} \tag{1.2}$$

*Received May 7, 2013; revised April 15, 2014. The authors sincerely thank Professor S. Peng for helpful discussions and suggestions. This work was partially supported by National College Students Innovation Training Project (48), the fund from NSFC (11301204), and the phD specialized grant of the Ministry of Education of China (20110144110001).

†Corresponding author.

where μ_j and $\beta_{jl} = \beta_{lj}$ are constants. These systems of equations, also known as Gross-Pitaevskii equations, have applications in many physical problems such as in nonlinear optics and in Bose-Einstein condensates theory for multi-species Bose-Einstein condensates. For more background of this problem, we can refer to [15] and the references therein. We should point out that the sign of the scattering length β determines whether the interactions of states are repulsive or attractive. In the attractive case the components of a vector solution tend to go along with each other, leading to synchronization. In the repulsive case, the components tend to segregate from each other, leading to phase separations. These phenomena have been documented in experiments as well as in numeric simulations [16].

Systems of nonlinear Schrödinger equations have been studied extensively by many mathematicians, for example, [2, 3, 7, 10, 12, 14, 24]. Phase separation was proved in several cases with constant potential, such as in [3, 5, 6, 14, 17, 19, 20] as the coupling constant β tends to negative infinity in the repulsive case. For the totally symmetric case ($\mu_j = \mu > 0$ for all j, and $\beta_{kj} = \beta$ for all $k \neq j$), in [17] Terracini and Verzini used variational methods and perturbation methods to construct radial solutions with domain separations for N-systems. In [6, 20], the minimax methods were applied to obtain infinity many radial positive solutions for 2-systems (see also [18] for generalizations to the N-systems). In [3], Bartsch, Dancer, and Wang used global bifurcation methods to obtain segregated radial solutions in repulsive cases for the general systems (1.1), establishing the existence of infinity branches of radial solutions with the property that $\sqrt{\mu-\beta}u - \sqrt{\nu-\beta}v$ has exactly k nodal domains for solutions along the kth branch. However, non-radial solutions of the segregated type with an arbitrarily large number of nodal domains are not well known. In [11], Lin and Wei obtained solutions with one component peaking at the origin and the other having a finite number of peaks on a k-polygon. For the symmetric case ($\mu = \nu$ and $P = Q \equiv 1$), in [20] Wei and Weth obtained infinitely many non-radial positive solutions for $\beta \leq -1$, which are potentially of the segregated type.

Very recently, in [15], applying techniques in singularly perturbed elliptic problems, Peng and Wang obtained infinitely many solutions with a potentially large number of nodal domains and constructed infinitely many non-radial synchronized and segregated solutions in $\mathbb{R}^3$. Both of the synchronized and segregated solutions have a large number bumps near infinity. Moreover, the locations of the bumps for u and v are roughly the same for the synchronized solutions and the locations of the bumps for u and v gave an angular shift for segregated solutions. In Remark 4.1 of [15], Peng and Wang mentioned that they did not know whether or not their result on segregated solutions was valid for $\mathbb{R}^2$. Inspired by this interesting problem, in this article, we intend to investigate existence of infinitely many positive segregated solutions for (1.1) in $\mathbb{R}^2$.

We assume that $P(r) > 0$ and $Q(r) > 0$ satisfy the following conditions:

(P) There are constants $a \in \mathbb{R}$, $m > 1$, and $\theta > 0$, such that as $r \to +\infty$,

$$P(r) = 1 + \frac{a}{r^m} + O\Big(\frac{1}{r^{m+\theta}}\Big). \tag{1.3}$$

(Q) There are constants $b \in \mathbb{R}$, $n > 1$, and $\epsilon > 0$, such that as $r \to +\infty$,

$$Q(r) = 1 + \frac{b}{r^n} + O\Big(\frac{1}{r^{n+\epsilon}}\Big). \tag{1.4}$$

Our main result in this article can be stated as follows: