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A STUDY ON THE BOUNDARY LAYER FOR THE PLANAR MAGNETOHYDRODYNAMICS SYSTEM*

Dedicated to Professor Tai-Ping Liu on the occasion of his 70th birthday

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Abstract The paper aims to estimate the thickness of the boundary layer for the planar MHD system with vanishing shear viscosity μ. Under some conditions on the initial and boundary data, we show that the thickness is of the order $\sqrt{\mu}|\ln\mu|$. Note that this estimate holds also for the Navier-Stokes system so that it extends the previous works even without the magnetic effect.

Key words magnetohydrodynamics system; boundary layer; vanishing shear viscosity

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1 Introduction

Consider the planar magnetohydrodynamics system

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \Big(\rho u^2 + p + \dfrac{1}{2}|\mathbf{b}|^2\Big)_x = \lambda u_{xx}, \\ (\rho \mathbf{w})_t + (\rho u \mathbf{w} - \mathbf{b})_x = \mu \mathbf{w}_{xx}, \\ \mathbf{b}_t + (u\mathbf{b} - \mathbf{w})_x = \nu \mathbf{b}_{xx}, \\ (\rho e)_t + (\rho e u)_x - (\kappa\theta_x)_x + p u_x = \lambda u_x^2 + \mu|\mathbf{w}_x|^2 + \nu|\mathbf{b}_x|^2 =: \mathcal{Q}, \end{cases} \tag{1.1}$$

where ρ denotes the density of the flow, θ the temperature, $u \in \mathbb{R}$ the longitudinal velocity, $\mathbf{w} \in \mathbb{R}^2$ the transversal velocity, $\mathbf{b} \in \mathbb{R}^2$ the transversal magnetic field, $p = p(\rho,\theta)$ the pressure and $e = e(\rho,\theta) > 0$ the internal energy, respectively. Moreover, $\kappa = \kappa(\rho,\theta)$ is the heat

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conductivity coefficient, and the positive constants λ, μ and ν represent the bulk viscosity, shear viscosity and the magnetic diffusivity coefficients, respectively. In this paper, we consider the perfect gas with the equation of state given by

$$p = R\rho\theta, \qquad e = C_v\theta, \tag{1.2}$$

with physical constants $R > 0$ and $C_v > 0$. Without loss of generality, set $C_v = 1$. Motivated by some physical models, such as the Boltzmann collision operator, assume κ depends only on θ as

$$\kappa = \kappa(\theta) = \theta^q, \quad q > 0. \tag{1.3}$$

In this paper, we consider the system (1.1) in a bounded domain $Q_T = \Omega \times (0,T)$ with $\Omega = (0,1)$ under the following initial and boundary conditions:

$$\begin{cases} (\rho, u, \theta, \mathbf{w}, \mathbf{b})(x,0) = (\rho_0, u_0, \theta_0, \mathbf{w}_0, \mathbf{b}_0)(x), \\ (u, \mathbf{b}, \theta_x)|_{x=0,1} = \mathbf{0}, \\ \mathbf{w}(0,t) = \mathbf{w}_1(t), \quad \mathbf{w}(1,t) = \mathbf{w}_2(t), \end{cases} \tag{1.4}$$

and try to understand the thickness of the boundary layer when the shear viscosity μ vanishes.

For this, let us first review some of the related works. First of all, the MHD system has been extensively studied because of its physical importance and mathematical difficulties, cf. [1, 2, 4, 11, 14, 15, 22–24] and the references therein.

Without the boundary effect, Vol'pert and Hudjaev [24] proved the existence and uniqueness of local solutions to this system. Some results on the system with small initial data were obtained in [10, 17, 19, 20]. When the heat conductivity coefficient κ is of the order of θ^q for some $q > 0$, some global existence solutions to the system (1.1) with large initial data were studied in [2, 3, 21] for $q \geq 2$, in [5] for $q \geq 1$ and in [6] for $q > 0$. On the other hand, for the case $q = 0$, the problem on the global existence of smooth solution to (1.1)–(1.4) with large initial data remains unsolved even though the corresponding problem for the Navier-Stokes equations was solved in [13] long time ago.

For problem in a bounded domain, the presence of boundary layer is a fundamental problem in fluid dynamics that can be traced back to the seminal work by Prandtl in 1904. For this, some results on the vanishing shear viscosity for the Navier-Stokes equations can be found in [7–9, 12, 19, 25] and the references therein. With the effect of magnetic field, the vanishing shear viscosity for the planar flow was studied in [5, 6] under the following condition on κ:

$$C^{-1}(1+\theta^q) \leq \kappa \equiv \kappa(\theta) \leq C(1+\theta^q)\,(q>0),\ \text{or,}\ \kappa \equiv \kappa(\rho) \geq C/\rho, \tag{1.5}$$

that avoids the degeneracy of θ around zero.

Without the magnetic effect, Frid and Shelukhin in [8] investigated the boundary layer of the compressible isentropic Navier-Stokes equations with cylindrical symmetry, and estimated the thickness of boundary layer (cf. Definition 1.1 below) in the order of $O(\mu^\alpha)(0 < \alpha < \frac{1}{2})$. For the non-isentropic Navier-Stokes equations, by imposing the following assumption on κ:

$$C^{-1}(1+\theta^q) \leq \kappa(\rho,\theta) \leq C(1+\theta^q), \quad |\kappa_\rho(\rho,\theta)| \leq C(1+\theta^q) \quad (q>1), \tag{1.6}$$

Jiang and Zhang in [12] obtained the same thickness estimate. In a recent paper [18], we improved these results to remove the constraint on the heat conductivity coefficient.