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TWENTY-EIGHT YEARS WITH "HYPERBOLIC CONSERVATION LAWS WITH RELAXATION"*

Dedicated to Professor Tai-Ping Liu on the occasion of his 70th birthday

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Abstract This paper is a review on the results inspired by the publication "Hyperbolic conservation laws with relaxation" by Tai-Ping Liu [1], with emphasis on the topic of nonlinear waves (specifically, rarefaction and shock waves). The aim is twofold: firstly, to report in details the impact of the article on the subsequent research in the area; secondly, to detect research trends which merit attention in the (near) future.

Key words conservation laws; relaxation; nonlinear waves; stability

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In 1987, Communications in Mathematical Physics published "Hyperbolic conservation laws with relaxation" by Tai-Ping Liu, [1]. After twenty-eight years, the paper trespassed the enviable amount of three hundreds citations (as revealed by combining the results in the databases MathScinet, Scopus and World of Science). Even if no bibliometric criterium is able to assess the importance of a scientific contribution, in the case under study, such result is coherent with the many qualities which makes of [1] an unmissable classic in conservation laws. Here, the intent is to take stock of the impact of the article and to freshen the attention on the research trends pointed out by it, many of which still deserve investigation.

1 The Whys and Wherefores

Three main facts contributed to make of [1] an exemplary research paper: the relevance of the mechanism incorporated in the class of conservation laws, the (nonlinear) validation of a readable stability condition, the subsequent publication of influential complementary papers.

1.1 Relaxation Structure

"Hyperbolic conservation law with relaxations" deals with systems of the form

$$\partial_t u + \partial_x f(u, v) = 0, \qquad \partial_t v + \partial_x g(u, v) = h(u, v), \tag{1.1}$$

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where the unknown (u,v) is a function of $(x,t) \in \mathbb{R} \times \mathbb{R}$. Hyperbolicity (and thus local well-posedness of the corresponding Cauchy problem) is guaranteed by the requirement that the jacobian $\partial(f,g)/\partial(u,v)$ has real eigenvalues. In [1], both u and v are scalars and there are two eigenvalues, $\lambda_{1,2} = \lambda_{1,2}(u,v)$ which determine the characteristic speeds of the system. For simplicity, the attention can be restricted to the strictly hyperbolic case, meaning that $\lambda_1 < \lambda_2$ for any (u,v) under consideration.

The relaxation structure is determined by the term h, which is chosen so that the dynamics leads (u,v) toward an equilibrium manifold $\mathcal{E}_*$, assumed to be the graph of a function v_*:

$$\mathcal{E}_* := \{(u,v) \,:\, h(u,v) = 0\} = \{(u,v) \,:\, v = v_*(u)\}.$$

The reduced equation, obtained from (1.1) by substituting the dynamic equation for v with the relation $v = v_*(u)$, is called equilibrium (or relaxed) equation,

$$\partial_t u + \partial_x f_*(u) = 0, \qquad \text{where} \quad f_*(u) := f\big(u, v_*(u)\big). \tag{1.2}$$

The requirement that the equilibrium manifold $\mathcal{E}_*$ is globally attracting for the underlying space independent kinetics is related to the sign of h and it is guaranteed if

$$\partial_v h(u,v) < 0. \tag{1.3}$$

A typical form for the function h is $(v_*(u) - v)/\tau$ where the relaxation time $\tau > 0$ determines the time-scale for the relaxation mechanism. Generalizations can be obtained by considering vectorial unknowns u and v, the former describing conserved quantities, the latter relaxation variables. If u is not scalar, it is crucial to determine the nature of the equilibrium equation (1.2), whose hyperbolicity is not guaranteed without specific assumptions.

Relaxation phenomena are ubiquitous in Applied Mathematics, relevant fields being the modelling of visco-elastic materials, and the study of traffic flows. Notable are the discrete-velocity reductions of the Boltzmann equation –such as the Broadwell model– which are semi-linear hyperbolic systems fitting into the family of conservation laws with relaxation.

The basic structure of relaxation models, as in the Extended Thermodynamics approach, presumes the presence of a reduced equation (or system) describing the dynamics close to an equilibrium configuration and an enlarged system dictating that, in the out-of-equilibrium regime, fluctuations are damped out by the relaxation mechanism. System (1.1) encompasses many of the basic features of such an extensive setting and, at the same time, is amenable of a rigorous approach in the hope of gaining insights valid in the general case.

1.2 Subcharacteristic Condition

Before [1], relaxation effects had already been explored in the milestone treatise [2]. Specifically, in Chapter 10, Whitham discusses, by means of relevant examples for gas-dynamics and traffic flows modelling, the role of the relation between the characteristic speeds λ_1, λ_2 of the principal part of (1.1) and the equilibrium speed

$$\lambda_* = \lambda_*(u) = f_*'(u) = \partial_u f - \partial_v f\, \partial_v h^{-1} \partial_u h\Big|_{\mathcal{E}_*} \tag{1.4}$$

of the equation (1.2), detecting the fundamental task played by the subcharacteristic condition

$$\lambda_1(u,v) < \lambda_*(u) < \lambda_2(u,v). \tag{1.5}$$