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AN ESTIMATE FOR THE MEAN CURVATURE OF SUBMANIFOLDS CONTAINED IN A HOROBALL*

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Abstract We obtain the Omori-Yau maximum principle on complete properly immersed submanifolds with the mean curvature satisfying certain condition in complete Riemannian manifolds whose radial sectional curvature satisfies some decay condition, which generalizes our previous results in [17]. Using this generalized maximum principle, we give an estimate on the mean curvature of properly immersed submanifolds in $\mathbb{H}^n \times \mathbb{R}^\ell$ with the image under the projection on $\mathbb{H}^n$ contained in a horoball and the corresponding situation in hyperbolic space. We also give other applications of the generalized maximum principle.

Key words Omori-Yau maximum principle; proper; mean curvature; horoball; hyperbolic space

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1 Introduction

There is a well-known Calabi-Chern problem (see [2, 5]) on the extrinsic boundedness properties for minimal submanifolds in n-dimensional Euclidean space.

Many efforts have been made during the past years, and the research on the Calabi-Chern problem has made some progress (see [3, 6, 9, 13, 15, 19] et al.). For $\mathbb{R}^3$, Nadirashvili [13] gave an example of complete immersed bounded minimal surface in $\mathbb{R}^3$ and Jorge-Xavier [9] constructed nonplanar, minimal isometric immersions of complete surfaces between two parallel planes in $\mathbb{R}^3$. Later, Jorge-Xavier [10] and Jorge-Koutroufiotis [8] proved that if the immersed manifold has scalar curvature bounded from below, then there is no bounded minimal immersion.

In [11], Lluch study an analogue problem for the hyperbolic space $\mathbb{H}^n(-c)$ of constant sectional curvature $-c < 0$, and proved that there is no minimal immersion with its image contained in a horoball under the assumption that the scalar curvature of the immersed submanifold is bounded from below.

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A more ambitious conjecture is: A complete non-flat minimal hypersurface in $\mathbb{R}^n$ has an unbounded projection in every $(n-2)$-dimensional flat subspace. This is not true for immersed minimal surfaces in $\mathbb{R}^3$ by Jorge-Xavier's example mentioned above.

On the other hand, Colding-Minicozzi [6] showed that the situation is different for embedded minimal disks in $\mathbb{R}^3$, which is proper, whereas the Nadirashvili's example and Jorge-Xavier's example are not proper.

Recently, Alias-Bessa-Dajczer [1] showed that a complete minimal immersed hypersurface in $\mathbb{R}^n (n \geq 3)$ with bounded projection in a two dimensional subspace cannot be proper. Qiu-Xin [17] later gave an estimate on the mean curvature of properly immersed submanifolds with bounded projection in N_1 in the product manifold $N_1 \times N_2$, which implied that any k-dimensional complete minimal submanifold in $N_1^{n_1} \times N_2^{n_2} (k > n_2)$ with bounded projection in N_1 cannot be proper.

Inspired by Calabi-Chern problem, it is natural to consider the problem for properly immersed submanifolds from the viewpoint of [11] and [1], and we obtain the following results.

Theorem 1 Let $\psi : M^m \to \mathbb{H}^n(-c)$ be a proper isometric immersion of a complete Riemannian manifold with mean curvature vector H. If $\psi(M) \subset \mathcal{HB}$ for some horoball $\mathcal{HB}$ of $\mathbb{H}^n(-c)$, then

$$\sup_M |H| \geq m\sqrt{c}.$$

Theorem 2 Let $\psi : M^m \to \mathbb{H}^n(-c) \times \mathbb{R}^\ell$ be a proper isometric immersion of a complete Riemannian manifold with mean curvature vector H. Assume that $m > \ell$. If $\psi(M) \subset \mathcal{HB} \times \mathbb{R}^\ell$ for some horoball $\mathcal{HB}$ of $\mathbb{H}^n(-c)$, then

$$\sup_M |H| \geq (m - \ell)\sqrt{c}.$$

The analytic tool to prove the above theorems is the Omori-Yau maximum principle. Omori [15] firstly gave a maximum principle on a complete Riemannian manifold. Later, Cheng and Yau [4, 19] refined and simplified the argument under the assumption on Ricci curvature bounded from below. The curvature assumption could be relaxed to strong quadratic decay of Ricci curvature in [3, 18] and a more general condition in [7]. There is a general analytic version of the Omori-Yau maximum principle due to Pigola et al. [16] and Lima-Pessoa [12]. Based on them, in this paper, we give the Omori-Yau maximum principle on complete properly immersed submanifolds of complete Riemannian manifolds whose radial sectional curvature has certain decay in Section 2, which generalizes our previous results in [17].

In Section 3, we give several geometric applications of the Omori-Yau maximum principle, including the proof of the above two theorems.

2 The Generalized Maximum Principle on Submanifolds

In this section, we shall give the Omori-Yau's maximum principle which holds on complete properly immersed submanifolds with the mean curvature satisfying certain condition in some complete Riemannian manifolds.

Theorem 3 Let N^n be a complete Riemannian manifold and $\psi : M^m \to N^n$ be a proper isometric immersion of a complete Riemannian manifold with mean curvature vector H. Let