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LELONG-DEMAILLY NUMBERS IN TERMS OF CAPACITY AND WEAK CONVERGENCE FOR CLOSED POSITIVE CURRENTS*

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Abstract In this paper we give a new definition of the Lelong-Demailly number in terms of the C_T-capacity, where T is a closed positive current and C_T is the capacity associated to T. This derived from some esimate on the growth of the C_T-capacity of the sublevel sets of a weighted plurisubharmonic (psh for short) function. These estimates enable us to give another proof of the Demailly's comparaison theorem as well as a generalization of some results due to Xing concerning the characterization of bounded psh functions. Another problem that we consider here is related to the existence of a psh function v that satisfies the equality $C_T(K) = \int_K T \wedge (dd^c v)^p$, where K is a compact subset. Finally, we give some conditions on the capacity C_T that guarantees the weak convergence $u_k T_k \to uT$, for positive closed currents T, T_k and psh functions u_k, u.

Key words positive current; capacity

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1 Introduction

Let T be a closed positive current of bidimension (p,p) on a bounded domain Ω of $\mathbb{C}^n$. In this paper we continue the study of the capacity C_T associated to T begun in [6]. Following [6] this capacity is defined by

$$C_T(E) = C_T(E,\Omega) = \sup\left\{\int_E T \wedge (dd^c v)^p,\ v \in \mathrm{psh}(\Omega),\ 0 \le v \le 1\right\}$$

for any Borel set $E \subset \Omega$ (see [6] for the properties of C_T). In the second section of this paper, we establish a result on the growth of the C_T-capacity of the sublevel sets of a weighted psh function φ in terms of the growth of the Monge-Ampère measure $T \wedge (dd^c\varphi)^p$. We refer the reader to Demailly's book [7] concerning the definition of that measure and the introduction of the generalized Lelong number. We use the estimate of Theorem 2.1 in the special case when $\varphi = \log|z|$ to obtain that the capacity C_T of a ball of radius r can be compared with

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the projective mass of T. The motivation for this lies is a question given to me by E. Bedford at the international meeting of Toulouse in 2008 concerning the behaviour of the capacity C_T of an open ball. As another interesting consequence, we introduce a new expression of the generalized Lelong number of T with respect to φ in terms of the C_T-capacity of the sublevel set of φ. As a third application of Theorem 2.1, we deduce that the polar set of the function φ is T-pluripolar in Ω. Recall that a subset $A \subset \Omega$ is said to be T-pluripolar if it has a vanishing C_T-capacity i.e., $C_T(A, \Omega) = 0$. This definition leads to a naturel but difficult problem that is the characterization of the T-pluripolar sets of Ω. Let us stress that this problem is unsolved even when $T = dd^c v$, for bounded psh function v. However, in [6], we dealed with the case of currents of integration over analytic subsets. Next, we use the estimate obtained so far together with the monotonicity of C_T to give another proof of the important comparison theorem of Demailly [7]. Furthermore, we deduce an extension of some results of Xing [9] about the class of psh functions which are bounded near the boundary.

According to Bedford-Taylor's fundamental paper [1], the capacity of a compact set K is equals to the Monge-Ampère measure of K, i.e., $C_{BT}(K) = \int_K (dd^c u_K^\star)^n$, where C_{BT} is the Bedford-Taylor capacity and $u_K^\star$ is the upper semicontinuous of the extremal function associated to K. This result had found quite a number of substantial applications for the developement of the pluripotential theory during the last 25 years. In Section 3, we focus our attention to a more general problem: given a positive closed current T of bidimension (p, p), can we find a bounded psh function v on Ω realizing the equality $C_T(K) = \int_K T \wedge (dd^c v)^p$? Notice that the study of such problem is already non-trivial for general positive closed T. However, when T is the current of integration of an analytic subset X of pure dimension p, the problem is reduced to the Bedford-Taylor's theory on the regular set of X, see [6]. In this paper, we restrict ourselves to the class of closed positive currents T that satisfy the following condition: every pluripolar set is T-pluripolar.

Let $(T_k)_k$ be a sequence of closed positive currents converging weakly to T in Ω and u_k, u are psh bounded functions in Ω such that u_k converges to u in C_T-capacity on each $E \Subset \Omega$. In the fourth section of this paper and under a suitable growth conditions on the mass of T_k with respect to C_T, we study the weak convergence $T_k \wedge dd^c u_k \to T \wedge dd^c u$. Recall that the problem of defining the wedge product $T \wedge dd^c u$ for psh function u has been investigated in several papers, e.g. [1], [4], [5], [7] and [8]. In [1], Bedford and Taylor proved that the complex Monge-Ampère operator $T \wedge (dd^c .)^p$ is continuous under monotone limits for bounded psh functions. Later, Demailly [7] generalized this result for psh functions which are bounded only near the intersection of the support of T with the boundary of Ω. Using the Demailly's hypothesis and the basic properties of the local potential associated to a positive closed current, Ben Messaoud and El Mir [4] obtained several convergence theorems. Also, it is worth noting that in [6] we generalized the Bedford-Taylor's result for locally uniformly bounded psh functions u_k which converge to a psh function u in the capacity C_T. We recall that a sequence of functions $(u_k)_k$ is said to converge to a function u in C_T-capacity on a set $E \subset \Omega$, if for each constant $\delta > 0$ we have

$$C_T\big(\{z \in E;\ |u_k - u| \geq \delta\}\big) \to 0, \quad \text{as } k \to \infty.$$

In the opposite direction, using the local expression of a closed positive current T in terms of its potential (see [4]), we give sufficient conditions that ensuring the convergence in capacity C_T of