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HARMONIC OSCILLATORS AT RESONANCE, PERTURBED BY A NON-LINEAR FRICTION FORCE*

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Abstract This note is an addendum to the results of Lazer and Frederickson [1], and Lazer [4] on periodic oscillations, with linear part at resonance. We show that a small modification of the argument in [4] provides a more general result. It turns out that things are different for the corresponding Dirichlet boundary value problem.

Key words resonance; existence of periodic solutions

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1 Introduction

We are interested in the existence of 2π periodic solutions to the problem ($x = x(t)$)

$$x'' + f(x)x' + n^2 x = e(t). \tag{1.1}$$

Here $e(t) \in C(R)$ satisfies $e(t + 2\pi) = e(t)$ for all t, $f(u) \in C(R)$, $n \geq 1$ is an integer. The linear part, $x'' + n^2 x = e(t)$, is at resonance, with the null space spanned by $\cos nt$ and $\sin nt$. Define $F(x) = \int_0^x f(t)\mathrm{d}t$. We assume that the finite limits $F(\infty)$ and $F(-\infty)$ exist, and

$$F(-\infty) < F(x) < F(\infty) \quad \text{for all } x. \tag{1.2}$$

Define

$$A_n = \int_0^{2\pi} e(t)\cos nt\mathrm{d}t, \quad B_n = \int_0^{2\pi} e(t)\sin nt\mathrm{d}t.$$

The following theorem was proved in case $n = 1$ by Lazer [4], based on Frederickson and Lazer [1]. The paper [1] was the precursor to the classical works of Landesman and Lazer [3], and Lazer and Leach [5].

Theorem 1.1 The condition

$$\sqrt{A_n^2 + B_n^2} < 2n\left(F(\infty) - F(-\infty)\right) \tag{1.3}$$

is necessary and sufficient for the existence of 2π periodic solution of (1.1).

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We provide a proof for all n, by modifying the argument in [4].

Remarkably, things are different for the corresponding Dirichlet boundary value problem, for which we derive a necessary condition for the existence of solutions, but show by a numerical computation that this condition is not sufficient.

2 The Proof

The following elementary lemmas are easy to prove.

Lemma 2.1 Consider a function $\cos(nt-\varphi)$, with an integer n and any real φ. Denote $P=\{t\in(0,2\pi)\,|\,\cos(nt-\varphi)>0\}$ and $N=\{t\in(0,2\pi)\,|\,\cos(nt-\varphi)<0\}$. Then

$$\int_P \cos(nt-\varphi)\mathrm{d}t=2,\qquad \int_N \cos(nt-\varphi)\mathrm{d}t=-2.$$

Lemma 2.2 Consider a function $\sin(nt-\varphi)$, with an integer n and any real φ. Denote $P_1=\{t\in(0,2\pi)\,|\,\sin(nt-\varphi)>0\}$ and $N_1=\{t\in(0,2\pi)\,|\,\sin(nt-\varphi)<0\}$. Then

$$\int_{P_1} \sin(nt-\varphi)\mathrm{d}t=2,\qquad \int_{N_1} \sin(nt-\varphi)\mathrm{d}t=-2.$$

Proof of the Theorem 1.1 1) Necessity

Given arbitrary numbers a and b, we can find a $\delta\in[0,2\pi)$, so that

$$a\cos nt+b\sin nt=\sqrt{a^2+b^2}\cos(nt-\delta)$$

($\cos\delta=\frac{a}{\sqrt{a^2+b^2}}$, $\sin\delta=\frac{b}{\sqrt{a^2+b^2}}$). We multiply (1.1) by $a\cos nt$, then by $b\sin nt$, integrate and add the results

$$I\equiv\int_0^{2\pi}F(x(t))'\cos(nt-\delta)\mathrm{d}t=\frac{aA_n+bB_n}{\sqrt{a^2+b^2}}. \tag{2.1}$$

Using that $x(t)$ is a 2π periodic solution, and Lemma 2.2, we have

$$I=n\int_0^{2\pi}F(x(t))\sin(nt-\delta)\mathrm{d}t=n\int_{P_1}+n\int_{N_1}<2n\left(F(\infty)-F(-\infty)\right).$$

Similarly, $I>-2n\left(F(\infty)-F(-\infty)\right)$, and so $|I|<2n\left(F(\infty)-F(-\infty)\right)$. On the right in (2.1) we have the scalar product of the vector (A_n,B_n) and an arbitrary unit vector. The condition (1.3) follows.

2) Sufficiency

We write our equation $(x'+F(x))'+n^2x=e(t)$ in the system form

$$x'=-F(x)+y,\qquad y'=-n^2x+e(t). \tag{2.2}$$

Setting $x=\frac{1}{n}X$, $y=Y$, we get

$$X'=-nF(\frac{1}{n}X)+nY,\qquad Y'=-nX+e(t). \tag{2.3}$$

Let $r(t)=\sqrt{X^2(t)+Y^2(t)}$. Then

$$r'(t)=\frac{XX'+YY'}{r(t)}=\frac{-nXF(\frac{1}{n}X)+e(t)Y}{r(t)}. \tag{2.4}$$

We see that if $r(t)$ is large, $r'(t)$ is bounded. It follows that there exists $r_0>0$, so that if $|r(0)|>r_0$, then $r(t)>0$ for all $t\in[0,2\pi]$, thus avoiding a singularity in (2.4). Switching to