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ON SOLVABILITY OF A BOUNDARY VALUE PROBLEM FOR A NONHOMOGENEOUS BIHARMONIC EQUATION WITH A BOUNDARY OPERATOR OF A FRACTIONAL ORDER*

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Abstract This paper is concerned with the solvability of a boundary value problem for a nonhomogeneous biharmonic equation. The boundary data is determined by a differential operator of fractional order in the Riemann-Liouville sense. The considered problem is a generalization of the known Dirichlet and Neumann problems.

Key words biharmonic equation; boundary value problem; fractional derivative; the Riemann-Liouville operator

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1 Introduction

Biharmonic equations appear in the study of mathematical models in several real-life processes as, among others, radar imaging [1] or incompressible flows [2].

Omitting a huge amount of works devoted to the study of this kind of equations, we refer some of them regarding to their used methods. Difference schemes and variational methods were used in the works [3, 4]. By using numerical and iterative methods, Dirichlet and Neumann boundary problems for biharmonic equations were studied in the papers [5, 6].

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There are some works, for example [7], where a computational method, based on the use of Haar wavelets was used for solving 2D and 3D Poisson and biharmonic equations. We also point out the work made in [8], where regularity of solutions for nonlinear biharmonic equations was investigated. In [9] and the dissertation [10] various problems for complex biharmonic and polyharmonic equations were investigated.

Along the paper we refer to the domain $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, as the unit ball. The dimension of the space is $n \geq 3$ and it is denoted $\partial\Omega = \{x \in \mathbb{R}^n : |x| = 1\}$ as the unit sphere. The usual Euclidean norm is written as $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$.

Now, for any $u : \Omega \to \mathbb{R}$ smooth enough function and a given $\alpha > 0$, denoting by $r = |x|$ and $\theta = x/|x|$, the appropriate integral operator of order α in the Riemann-Liouville can be defined, in a similar sense to [11, p.69], by the following expression

$$J^\alpha[u](x) = \frac{1}{\Gamma(1-\alpha)} \int_0^r (r-\tau)^{\alpha-1} u(\tau\theta)\mathrm{d}\tau, \quad x \in \Omega.$$

In what follows, we suppose $J^0[u](x) = u(x)$ for all $x \in \Omega$.

Let $m - 1 < \alpha \leq m$ for some $m = 1, 2, \cdots$. The Riemann-Liouville derivative of order α is defined as, see [11, p.70],

$$D^\alpha[u](x) = \frac{\partial^m}{\partial r^m} J^{m-\alpha}[u](x), \quad x \in \Omega,$$

where $\frac{\partial}{\partial r}$ is the differential operator

$$\frac{\partial}{\partial r} = \frac{1}{r} \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}, \quad \frac{\partial^k u}{\partial r^k} = \frac{\partial}{\partial r}\left(\frac{\partial^{k-1} u}{\partial r^{k-1}}\right), \quad k = 1, 2, \cdots.$$

Now, we introduce the following functionals:

$$B^\alpha[u](x) = r^\alpha D^\alpha[u](x), \ m - 1 < \alpha \leq m, \quad m = 1, 2, \cdots$$

and

$$B^{-\alpha}[u](x) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} s^{-\alpha} u(sx)\mathrm{d}s, \quad 0 < \alpha \leq 1.$$

It should be noted that some properties of the previous operators are studied for $0 < \alpha < 1$ in the work [12]. There it is proved that these operators are mutually inverse in the class of harmonic functions for $0 < \alpha < 1$.

2 Statement of the Problem and Formulation of the Main Result

Let $0 < \alpha \leq 1$ be fixed and consider the following boundary value problem

$$\begin{cases} \Delta^2 u(x) = f(x), & x \in \Omega, & (1) \\ B^\alpha[u](x) = \varphi_1(x), & x \in \partial\Omega, & (2) \\ B^{\alpha+1}[u](x) = \varphi_2(x), & x \in \partial\Omega, & (3) \end{cases}$$

where f, φ_1 and φ_2 are given functions. A solution of the problem (1)–(3) will be a function $u \in C^4(\Omega) \cap C(\bar{\Omega})$ such that $B^{\alpha+k}[u] \in C(\overline{\Omega})$, $k = 0, 1$, and satisfies the equation (1) coupled with the boundary conditions (2) and (3).