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OPTIMAL SUMMATION INTERVAL AND NONEXISTENCE OF POSITIVE SOLUTIONS TO A DISCRETE SYSTEM*

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Abstract In this paper, we are concerned with properties of positive solutions of the following Euler-Lagrange system associated with the weighted Hardy-Littlewood-Sobolev inequality in discrete form

$$\begin{cases} u_j = \displaystyle\sum_{k\in\mathbb{Z}^n} \frac{v_k^q}{(1+|j|)^\alpha(1+|k-j|)^\lambda(1+|k|)^\beta}, \\ v_j = \displaystyle\sum_{k\in\mathbb{Z}^n} \frac{u_k^p}{(1+|j|)^\beta(1+|k-j|)^\lambda(1+|k|)^\alpha}, \end{cases} \tag{0.1}$$

where $u, v > 0, 1 < p, q < \infty, 0 < \lambda < n, 0 \le \alpha+\beta \le n-\lambda, \frac{1}{p+1} < \frac{\lambda+\alpha}{n}$ and $\frac{1}{p+1} + \frac{1}{q+1} \le \frac{\lambda+\alpha+\beta}{n} := \frac{\bar{\lambda}}{n}$. We first show that positive solutions of (0.1) have the optimal summation interval under assumptions that $u \in l^{p+1}(\mathbb{Z}^n)$ and $v \in l^{q+1}(\mathbb{Z}^n)$. Then we show that problem (0.1) has no positive solution if $0 < pq \le 1$ or $pq > 1$ and $\max\{\frac{(n-\bar{\lambda})(q+1)}{pq-1}, \frac{(n-\bar{\lambda})(p+1)}{pq-1}\} \ge \bar{\lambda}$.

Key words summation; optimal interval; nonexistence; weighted Hardy-Littlewood-Sobolev inequality

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1 Introduction

In this paper, we investigate the summation and nonexistence of the positive solutions to the following Euler-Lagrange system associated with the weighted Hardy-Littlewood-Sobolev inequality in discrete form

$$\begin{cases} u_j = \displaystyle\sum_{k\in\mathbb{Z}^n} \frac{v_k^q}{(1+|j|)^\alpha(1+|k-j|)^\lambda(1+|k|)^\beta}, \\ v_j = \displaystyle\sum_{k\in\mathbb{Z}^n} \frac{u_k^p}{(1+|j|)^\beta(1+|k-j|)^\lambda(1+|k|)^\alpha}, \end{cases} \tag{1.1}$$

where $u, v > 0, 1 < p, q < \infty, 0 < \lambda < n, \frac{1}{p+1} < \frac{\lambda+\alpha}{n}$ and

$$\frac{1}{p+1} + \frac{1}{q+1} \le \frac{\lambda+\alpha+\beta}{n}. \tag{1.2}$$

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Problem (1.1) is related to the weighted Hardy-Littlewood-Sobolev inequality in discrete form:

$$\left|\sum_{j\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n}\frac{f_jg_k}{(1+|j|)^\alpha(1+|k-j|)^\lambda(1+|k|)^\beta}\right|\le C_{\alpha,\beta,\lambda,n}\|f\|_r\|g\|_s, \tag{1.3}$$

where $\|f\|_r=\Big(\sum\limits_{j\in\mathbb{Z}^n}|f_j|^r\Big)^{1/r}$, $r,s>1, 0<\lambda<n, 0\le\alpha+\beta<n-\lambda$ and the powers of the weights satisfy

$$\frac{1}{r}+\frac{1}{s}+\frac{\lambda+\alpha+\beta}{n}\ge 2. \tag{1.4}$$

To find the best constant in (1.3), one can maximize the functional

$$J(f,g)=\sum_{j\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n}\frac{f_jg_k}{(1+|j|)^\alpha(1+|k-j|)^\lambda(1+|k|)^\beta} \tag{1.5}$$

under the constraints $\|f\|_r=\|g\|_s=1$. Then we obtain the systems of Euler-Lagrange equations:

$$\begin{cases}\lambda_1 rf_j^{r-1}=\displaystyle\sum_{k\in\mathbb{Z}^n}\frac{g_k}{(1+|j|)^\alpha(1+|k-j|)^\lambda(1+|k|)^\beta},\\ \lambda_2 sg_j^{s-1}=\displaystyle\sum_{k\in\mathbb{Z}^n}\frac{f_k}{(1+|j|)^\beta(1+|k-j|)^\lambda(1+|k|)^\alpha},\end{cases} \tag{1.6}$$

where $\lambda_1 r=\lambda_2 s=J(f,g)$.

Let $u_j=c_1f_j^{r-1}, v_j=c_2g_j^{s-1}, p=\frac{1}{r-1}, q=\frac{1}{s-1}$, when $pq\ne 1$, (1.6) turns into

$$\begin{cases}u_j=\displaystyle\sum_{k\in\mathbb{Z}^n}\frac{v_k^q}{(1+|j|)^\alpha(1+|k-j|)^\lambda(1+|k|)^\beta},\\ v_j=\displaystyle\sum_{k\in\mathbb{Z}^n}\frac{u_k^p}{(1+|j|)^\beta(1+|k-j|)^\lambda(1+|k|)^\alpha},\end{cases} \tag{1.7}$$

where $0<p,q<\infty, 0<\lambda<n, \alpha+\beta\ge 0, \frac{1}{p+1}<\frac{\lambda+\alpha}{n}$ and $\frac{1}{p+1}+\frac{1}{q+1}\le\frac{\lambda+\alpha+\beta}{n}:=\frac{\bar\lambda}{n}$.

In the special case when $\alpha=\beta=0$, inequality (1.3) reduces to

$$\sum_{j\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n}\frac{f_jg_k}{(1+|k-j|)^\lambda}\le C_{\lambda,n}\|f\|_r\|g\|_s, \tag{1.8}$$

which can be written in another form

$$\sum_{j\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n,k\ne j}\frac{f_jg_k}{|k-j|^\lambda}\le C'_{\lambda,n}\|f\|_r\|g\|_s. \tag{1.9}$$

When $n=1$ in (1.9) and $r,s>1, \frac{1}{r}+\frac{1}{s}>1, \lambda=2-(\frac{1}{r}+\frac{1}{s})$, then it is the Hardy-Littlewood-Pólya inequality, which can be found in [6], inequality 381, page 288.

Recently, Li and Villavert [13] extended the well-known Hardy-Littlewood-Pólya inequality in the case $p=q=2$ and $\lambda=1$ with a logarithm correction. While Cheng and Li [5] considered the more general case that $\lambda=n$ and $p=q=2$. They first obtain a sharp estimate for the best constant, then for the optimizer, they prove the uniqueness and a symmetry property. At the same time, Huang, Li and Yin [7] proved that the best constant in (1.9) can be achieved when $\frac{1}{r}+\frac{1}{s}+\frac{\lambda}{n}>2$. For the double weighted case, the minimize of (1.5) can be done in the same way when r and s are supercritical, that is $\frac{1}{r}+\frac{1}{s}+\frac{\lambda+\alpha+\beta}{n}>2$, and we guess that it is also true for the critical case, $\frac{1}{r}+\frac{1}{s}+\frac{\lambda+\alpha+\beta}{n}=2$.