Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2014,**34B**(6):1741–1748
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

GLOBAL REGULARITY FOR MODIFIED CRITICAL DISSIPATIVE QUASI-GEOSTROPHIC EQUATIONS*

Wanrong YANG (杨婉蓉)
Department of Mathematics, Beifang University of Nationalities, Ningxia 750021, China
E-mail: *yangwanrong1618@163.com*

Quansen JIU (酒全森)
Department of Mathematics, Capital Normal University, Beijing 100037, China
E-mail: *jiuqs@mail.cnu.edu.cn*

Abstract We consider the n-dimensional modified quasi-geostrophic (SQG) equations

$$\partial_t\theta + u\cdot\nabla\theta + \kappa\Lambda^\alpha\theta = 0,$$

$$u = \Lambda^{\alpha-1}R^\perp\theta$$

with $\kappa>0,\alpha\in(0,1]$ and $\theta_0\in W^{1,\infty}(\mathbb{R}^n)$. In this paper, we establish a different proof for the global regularity of this system. The original proof was given by Constantin, Iyer, and Wu [5], who employed the approach of Besov space techniques to study the global existence and regularity of strong solutions to modified critical SQG equations for two dimensional case. The proof provided in this paper is based on the nonlinear maximum principle as well as the approach in Constantin and Vicol [2].

Key words quasi-geostrophic equations; global regularity; maximum principle

2010 MR Subject Classification 35Q35; 76D03

1 Introduction

This paper studies the global regularity of the following modification of the n-dimensional dissipative quasi-geostrophic equations

$$\begin{cases}\partial_t\theta + u\cdot\nabla\theta + \kappa\Lambda^\alpha\theta = 0, & x\in\mathbb{R}^n,\ t>0,\\ u = \Lambda^{\alpha-1}R^\perp\theta, & x\in\mathbb{R}^n,\ t>0\end{cases} \tag{1.1}$$

where $n\geq 2,\kappa>0,\alpha\in(0,1]$. $\Lambda=(-\triangle)^{\frac{1}{2}}$ is the Zygmund operator, and

$$R^\perp\theta = \Lambda^{-1}(-\partial_2\theta,\partial_1\theta).$$

Not that when $\alpha=1$ this is the critical dissipative quasi-geostrophic equation. The case of $\alpha=0$ arises when θ is the vorticity of an n-dimensional damped inviscid incompressible

*Received September 22, 2013; revised May 18, 2014. Jiu was supported by Project of Beijing Chang Cheng Xue Zhe (11228102), and was also supported by NSF of China (11171229, 11231006).

fluid. When $\kappa > 0, \alpha \in (0,1)$, the dissipation term is the same as that of the supercritical quasi-geostrophic equation, however the extra $\Lambda^{\alpha-1}$ in the definition of u makes the drift term $u \cdot \nabla\theta$ scale the same way as the scaling $\theta_\epsilon(x,t) = \theta(\epsilon x, \epsilon^\alpha t)$, similar to the scaling invariance of the critical dissipative quasi-geostrophic equations.

For the two dimensional dissipative quasi-geostrophic equations

$$\begin{cases} \partial_t\theta + u \cdot \nabla\theta + \kappa\Lambda^\alpha\theta = 0, & x \in \mathbb{R}^2,\ t > 0, \\ u = R^\perp\theta, & x \in \mathbb{R}^2,\ t > 0, \end{cases} \tag{1.2}$$

the problem of global existence of smooth solutions has been extensively studied, partly because ones have emphasized a deep analogy between the two dimensional critical dissipative quasi-geostrophic equations and the three dimensional Navier-Stokes equations. Kiselev, Nazarov and Volberg [9] settled the global existence to (1.2) in the periodic case. Caffarelli-Vasseur [6] established the global existence of smooth solutions to (1.2) when $\alpha = 1$ by using De Giorgi method. In the supercritical case ($0 < \alpha < 1$) global existence of smooth solutions is still open. Constantin and Jiahong Wu ([3], [4]) extended the framework of Caffarelli-Vasseur [6] to the supercritical case ($0 < \alpha < 1$) by requiring additional assumptions: Hölder continuity of solutions.

However, the story appears to be different for the modified dissipative quasi-geostrophic equations (1.1). In this paper, we give a different proof of the global existences of smooth solutions to the n-dimensional modified critical dissipative quasi-geostrophic equations (1.1). The proof provided in this paper is based on the nonlinear maximum principle as well as the approach in Constantin and Vicol [2]. The original proof was given by Constantin, Iyer and Wu [5], who studied the global existence and regularity of strong solutions to (1.1) for two dimensional case by using the approach of Besov space techniques. Jiu, Miao, Wu and Zhang studied two-dimensional Boussinesq equations with general critical dissipation in [8] in which the nonlinear maximum principle was also applied. Moreover, in order to establish our main result, the whole proof is splitted in two steps. The first step shows that if a solution of the (1.1) has "only small shocks", then it is regular (cf. Proposition 2.1 below). The second step shows that if the initial data has only small shocks, then the solution has only small shocks for all later times (cf. Proposition 2.3 below). Our global existence result can be stated as follows.

Theorem 1.1 Assume the initial data $\theta_0 \in W^{1,\infty}(\mathbb{R}^n)$ and have sufficient decay at spacial infinity, then (1.1) has a unique global smooth solution $\theta(x,t)$ satisfying, for any $0 < T < \infty$, $\theta \in C^\infty((0,T) \times \mathbb{R}^n)$.

To give a precise statement of our result, we first provide the definition of only small shocks (OSS_δ).

Definition 1.2 Let $\delta > 0$, and $t > 0$. We say $\theta(x,t)$ has the OSS_δ property, if there exists an $L > 0$ such that

$$\sup_{(x,y):|x-y|<L} |\theta(x,t) - \theta(y,t)| \le \delta.$$

Moreover, for $T > 0$, we say $\theta(x,t)$ has the uniform OSS_δ property on $[0,T]$, if there exists an $L > 0$ such that

$$\sup_{(x,y):|x-y|<L,0\le t\le T} |\theta(x,t) - \theta(y,t)| \le \delta. \tag{1.3}$$