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SOME NECESSARY AND SUFFICIENT CONDITIONS FOR EXISTENCE OF POSITIVE SOLUTIONS FOR THIRD ORDER SINGULAR SUBLINEAR MULTI-POINT BOUNDARY VALUE PROBLEMS*

Zhongli WEI (韦忠礼)
Department of Mathematics, Shandong Jianzhu University, Jinan 250101, China
School of Mathematics, Shandong University, Jinan 250100, China
E-mail: *jnwzl32@163.com*

Abstract We mainly study the existence of positive solutions for the following third order singular multi-point boundary value problem

$$\begin{cases} x^{(3)}(t)+f(t,\ x(t),\ x'(t))=0, \quad 0<t<1, \\ x(0)-\displaystyle\sum_{i=1}^{m_1}\alpha_i x(\xi_i)=0, \quad x'(0)-\displaystyle\sum_{i=1}^{m_2}\beta_i x'(\eta_i)=0, \quad x'(1)=0, \end{cases}$$

where $0\le\alpha_i\le\sum\limits_{i=1}^{m_1}\alpha_i<1$, $i=1,\ 2,\cdots,m_1$, $0<\xi_1<\xi_2<\cdots<\xi_{m_1}<1$, $0\le\beta_j\le\sum\limits_{i=1}^{m_2}\beta_i<1$, $j=1,\ 2,\cdots,m_2$, $0<\eta_1<\eta_2<\cdots<\eta_{m_2}<1$. And we obtain some necessary and sufficient conditions for the existence of $C^1[0,1]$ and $C^2[0,1]$ positive solutions by constructing lower and upper solutions and by using the comparison theorem. Our nonlinearity $f(t,x,y)$ may be singular at $x,y,t=0$ and/or $t=1$.

Key words boundary value problems; positive solutions; lower and upper solutions; comparison theorem

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1 Introduction

Third order differential equations arise in a variety of different areas of applied mathematics and physics, as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves or gravity driven flows, and so on. The existence of positive solutions of third-order nonsingular boundary value problems has been studied by several authors. Using the Krasnoselskii', Leggett-Williams and five functions fixed-point theorems, the authors in [1]–[3] gave some sufficient conditions for the existence of multiple positive solutions. Yao and Feng in [4] used the method of upper and lower solutions to prove existence results for

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the following third-order two-point boundary value problem:

$$u^{(3)}(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad u(0) = u'(0) = u'(1) = 0.$$

By the use of a new maximum principle and the method of upper and lower solutions, Feng and Liu in [5] established existence results for a third-order two-point boundary value problem:

$$u^{(3)}(t) + f(t, u(t), u'(t)) = 0, \quad 0 < t < 1, \quad u(0) = u'(0) = u'(1) = 0.$$

Liu et al. in [6] used Krasnoselskii' fixed-point theorem to prove some existence results of singular boundary value problem:

$$u^{(3)}(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1, \quad u(a) = u(b) = u''(b) = 0.$$

Very recently, El-Shahed in [7] established the existence of positive solutions to boundary value problem:

$$u^{(3)}(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1, \quad u(0) = u'(0) = 0, \quad \alpha u'(1) + \beta u''(1) = 0,$$

where $\lambda,\ \alpha,\ \beta$ are positive parameters. Feng et al. in [8] used Krasnoselskii' fixed-point theorem to prove some existence results of the singular three-point boundary value problem:

$$u^{(3)}(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1, \quad u(0) = \alpha u'(0), \quad u(1) = \beta u(\eta), \quad u'(1) = 0,$$

where $0 < \eta < 1,\ \alpha,\ \beta > 0$, λ is a positive parameter. Function $f(u)$ in papers [4]–[8] cannot be singular at $u = 0$, and they only obtained some sufficient conditions for the existence of solutions of corresponding problems.

The theory of singular boundary value problems has become an important area of investigation in recent years (see [9]–[13] and the references therein). In this paper, we shall consider the positive solutions to the following nonlinear singular multi-point boundary value problems of third-order differential equation:

$$x^{(3)}(t) + f(t,\ x(t),\ x'(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$x(0) - \sum_{i=1}^{m_1} \alpha_i x(\xi_i) = 0, \quad x'(0) - \sum_{i=1}^{m_2} \beta_i x'(\eta_i) = 0, \quad x'(1) = 0, \tag{1.2}$$

where $0 \le \alpha_i \le \sum\limits_{i=1}^{m_1} \alpha_i < 1$, $i = 1,\ 2, \cdots, m_1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m_1} < 1$, $0 \le \beta_j \le \sum\limits_{i=1}^{m_2} \beta_i < 1$, $j = 1,\ 2, \cdots, m_2$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m_2} < 1$, and f satisfies the following hypothesis:

($\mathbb{H}$) $f \in C((0,1) \times (0,\ \infty) \times (0,\ \infty), [0,\ +\infty))$, there exist constants λ_i, μ_i $(-\infty < \lambda_i \le 0 \le \mu_i,\ i = 1, 2,\ \mu_1 + \mu_2 < 1,\ \lambda_2 < \mu_2)$ such that for $t \in (0,1), x, y \in (0, +\infty)$,

$$c^{\mu_1} f(t,x,y) \le f(t,cx,y) \le c^{\lambda_1} f(t,x,y), \quad \text{if} \ \ 0 < c \le 1, \tag{1.3}$$

$$c^{\mu_2} f(t,x,y) \le f(t,x,cy) \le c^{\lambda_2} f(t,x,y), \quad \text{if} \ \ 0 < c \le 1. \tag{1.4}$$

Remark 1.1 Obviously, (1.3) and (1.4) are equivalent to

$$c^{\lambda_1} f(t,x,y) \le f(t,cx,y) \le c^{\mu_1} f(t,x,y), \quad \text{if} \ \ c \ge 1, \tag{1.5}$$

$$c^{\lambda_2} f(t,x,y) \le f(t,x,cy) \le c^{\mu_2} f(t,x,y), \quad \text{if} \ \ c \ge 1. \tag{1.6}$$

By singularity we mean that the function f in (1.1) is allowed to be unbounded at the points $x = 0,\ y = 0,\ t = 0$ and / or $t = 1$. A function $x(t) \in C^1[0,1] \cap C^3(0,1)$ is called a