Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2014,**34B**(6):1907–1921
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

EXISTENCE RESULTS FOR DEGENERATE ELLIPTIC EQUATIONS WITH CRITICAL CONE SOBOLEV EXPONENTS*

Haining FAN (范海宁)†
School of Sciences, China University of Mining and Technology, Xuzhou 221116, China
Department of Mathematics, Wuhan University, Wuhan 430072, China
E-mail: *fanhaining888@163.com*

Xiaochun LIU (刘晓春)
Department of Mathematics, Wuhan University, Wuhan 430072, China
E-mail: *xcliu@whu.edu.cn*

Abstract In this paper, we study the existence result for degenerate elliptic equations with singular potential and critical cone sobolev exponents on singular manifolds. With the help of the variational method and the theory of genus, we obtain several results under different conditions.

Key words existence results; variational method; critical cone Sobolev exponent; singular potential

2010 MR Subject Classification 35J20; 58J05

1 Preliminaries

In this paper, we consider the existence of solutions for the following Dirichlet problem:

$$(E_{\alpha,\beta})\begin{cases} -\Delta_{\mathbb{B}}u-\mu V(x)u=\alpha|u|^{2^*-2}u+\beta f(x)|u|^{q-2}u, & x\in \text{int}\mathbb{B},\\ u=0, & x\in\partial\mathbb{B},\end{cases}$$

where $\mathbb{B}=[0,1)\times X$ for $X\subseteq\mathbb{R}^{N-1}$ bounded and $0\in X$. Here $\mu\in\mathbb{R},\alpha,\beta\geq 0$ and $1<q<2^*=2N/(N-2)(N\geq 3)$, where 2^* is the critical cone Sobolev exponent according to the continuous embedding $\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})\hookrightarrow L_{2^*}^{\frac{N}{2^*}}(\mathbb{B})$ (see [3] or Propositions 2.1–2.3 for details). The operator $\Delta_{\mathbb{B}}$ is defined by $(x_1\partial_{x_1})^2+\partial_{x_2}^2+\cdots+\partial_{x_N}^2$, which is an elliptic operator with totally characteristic degeneracy on the boundary $x_1=0$ (we also all it Fuchsian type Laplacian), and the corresponding gradient operator is denoted by $\nabla_{\mathbb{B}}:=(x_1\partial_{x_1},\partial_{x_2},\cdots,\partial_{x_N})$. Moreover, $V(x)$ is the so-called cone Hardy weight and $f(x)$ is a weight function defined on $\mathbb{B}$.

The starting point of the analysis of operators on manifolds with conical singularities was due to the work of Kondratev [11]. Then, many authors such as Schulze, Gil, Seiler, Krainer,

*Received April 23, 2013; revised October 30, 2013. This work was supported by NSFC (11371282).
†Corresponding author: Haining FAN.

Witt, Mazzeo, Melrose and Mendoza subsequently developed this subject by the tool of calculus of pseudo-differential operators on manifolds with singularities and they also gave various methods and ideas in the pseudo-differential calculus on manifolds with singularities (see [7, 15–17, 19, 20]). We would like to mention the progress for the existence theorems for the corresponding nonlinear elliptic equations on manifolds with conical singularities in recent years. For $\mu=\alpha=0$, $\beta=1$ and $f\equiv 1$, the authors in [3] considered the following equation:

$$\begin{cases} -\Delta_{\mathbb{B}}u=|u|^{q-2}u, & x\in \mathrm{int}\mathbb{B},\\ u=0, & x\in\partial\mathbb{B}, \end{cases}$$

and obtained a nontrivial solution if $2<q<2^*$. For $\mu=0$, $\alpha=1$, $q=2$ and $f\equiv 1$, the authors studied the following problem

$$(\overline{E})\begin{cases} -\Delta_{\mathbb{B}}u=|u|^{2^*-2}u+\beta u, & x\in \mathrm{int}\mathbb{B},\\ u=0, & x\in\partial\mathbb{B}, \end{cases}$$

in [2, 4, 12]. In [2, 4], they showed that equation admits at least a non-trivial solution when $N\geq 4$ and $\beta>0$ is small enough, and $(\overline{E})$ has infinitely many solutions under the condition $N\geq 7$. Furthermore, the authors in [12] studied the same problem and pointed out that $(\overline{E})$ possesses at least a nodal solution if $N\geq 4$ and $\beta>0$ is small enough. Similar problems can be found in [4, 5, 9, 10] and the references [2–9, 11, 12, 15–17, 19, 20].

The following is cone Hardy inequality, which was proved in [6]. There exists $C_1>0$ such that

$$\int_{\mathbb{B}} V(x)|u|^2\frac{\mathrm{d}x_1}{x_1}\mathrm{d}x'\leq C_1\int_{\mathbb{B}}|\nabla_{\mathbb{B}}u|^2\frac{\mathrm{d}x_1}{x_1}\mathrm{d}x' \text{ for all } u\in\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B}), \tag{1.1}$$

where $V(x)$ has two kinds of singular potential functions:

$$V_1=\left(\frac{n-3}{2}\right)^2\frac{1}{|x|^2},$$

and

$$V_2=\left(\frac{N-1}{2}\right)^2\frac{|x_1|^{-2}\mathrm{e}^{-1/|x_1|^2}}{\mathrm{e}^{-1/|x_1|^2}+|x_2|^2+\cdots+|x_N|^2}.$$

Clearly, 0 is the singular point of V_1 and V_2 is unbounded on $\partial\mathbb{B}$. Notice that V_1 is the classical Hardy potential for Δ and V_2 is an extra Hardy potential as a new discovery for $\Delta_{\mathbb{B}}$, and it is interesting to consider the existence of solutions of degenerate elliptic equations involving different kinds of Hardy potentials. We denote

$$\overline{\mu}=\inf_{u\in\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\mathbb{B})\setminus\{0\}}\frac{\int_{\mathbb{B}}|\nabla_{\mathbb{B}}u|^2\frac{\mathrm{d}x_1}{x_1}\mathrm{d}x'}{\int_{\mathbb{B}}V(x)|u|^2\frac{\mathrm{d}x_1}{x_1}\mathrm{d}x'}. \tag{1.2}$$

Based on (1.2), the author in [6] showed

$$\begin{cases} -\Delta_{\mathbb{B}}u-\mu V(x)u=\lambda u+|u|^{q-2}u, & x\in \mathrm{int}\mathbb{B},\\ u=0, & x\in\partial\mathbb{B} \end{cases}$$

has infinitely many solutions when $\lambda>0$, $0<\mu<\overline{\mu}$ and $2<q<2^*$.

Motivated by [2–9, 11, 12], we consider $(E_{\alpha,\beta})$ under several different conditions on q and μ in this paper. Moreover, we assume $f(x)\geq 0$ on $\mathbb{B}$ and satisfies