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SECTORIAL OSCILLATION THEORY OF LINEAR DIFFERENTIAL EQUATIONS*

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Abstract In this paper, the sectorial oscillation of the solutions of higher order homogeneous linear differential equations

$$f^{(k)}+A_{n-2}(z)f^{(k-2)}+\cdots+A_1(z)f'+A_0(z)f=0$$

with infinite order entire function coefficients is studied. Results are obtained to extend some results in [19] and [18].

Key words radial exponent of convergence; iterated order; sectorial oscillation; Borel direction

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1 Introduction and Main Results

In this paper, unless otherwise stated, we mean a meromorphic function that is defined and meromorphic in the whole complex plane $\mathbb{C}$. We use the standard notation of Nevalinna's value distribution theory and assume that the reader is familiar with the basic results and the standard notations of the Nevanlinna's value distribution theory (see [24] or [26]). Since 1982 when the article [1] appeared in Trans. Amer. Math. Soc., there have appeared many papers on the global theory of complex differential equations which were studied from the point of view of Nevanlinna theory. We refer the reader to the books by Laine [12], and Gao etc. [7]. The first general research on the sectorial oscillation theory of the solutions of

$$f''-A(z)f=0 \tag{1}$$

is due to Wang [16] and Wu [19], respectively. Here, we recall some definitions and research background by Wang [16] (also see Rossi and Wang [14]) and Wu [22] as follows.

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Definition 1 (sectorial exponent of convergence) For $a \in \mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$, define

$$\lambda_{\alpha,\beta}(f,a) = \limsup_{r\to\infty} \frac{\log n(r, X(\alpha,\beta), f=a)}{\log r},$$

where $X(\alpha,\beta) = \{z | \alpha < \arg z < \beta\}, 0 < \beta - \alpha \le \pi$ and $n(r, X(\alpha,\beta), f=a)$ is the number of the roots of $f(z) - a = 0$ in $X(\alpha,\beta) \cap \{|z| < r\}$, counting multiplicities. Especially, when $a=0$, we write $\lambda_{\alpha,\beta}(f) = \lambda_{\alpha,\beta}(f,0)$.

Definition 2 (radial exponent of convergence) For any $\theta \in [0, 2\pi)$ and $a \in \mathbb{C}_\infty$, we define

$$\lambda_\theta(f,a) = \lim_{\varepsilon\to 0^+} \lambda_{\theta-\varepsilon,\theta+\varepsilon}(a,f).$$

Especially, when $a=0$, we write $\lambda_\theta(f) = \lambda_\theta(f,0)$.

In 1994, Wu [19] proved the following theorem.

Theorem A [19] Let $A(z)$ be a transcendental entire function of finite order in the plane and let f_1, f_2 be two linearly independent solutions of (1). Set $E = f_1 f_2$. Then $\lambda_\theta(E) = +\infty$, if and only if $\limsup\limits_{r\to\infty} \frac{\log\log M(r, X_{\theta,\varepsilon}, E)}{\log r} = +\infty$ for any $\varepsilon > 0$, where $M(r, X_{\theta,\varepsilon}, E) = \sup\{|E(te^{i\tau})| : \theta - \varepsilon \le \tau \le \theta + \varepsilon, 1 \le t \le r\}$.

Recently, Wu [18] proved the following theorem on connection of the radial exponent of convergence of zeros with Borel direction of the product of a solution base of (1).

Theorem B [18] Let $A(z)$ be a transcendental entire function of finite order in the plane and f_1, f_2 be two linearly independent solutions of (1). Let $E = f_1 f_2$. Suppose that the exponent of convergence of zero-sequence $\lambda(E)$ is ∞. Then $L : \arg z = \theta_0$ is an infinity order Borel direction of E if and only if $\lambda_{\theta_0}(E) = \infty$.

In 2009, Wu, Tian and Wu [23] gave a new proof of Theorem B. For $k \ge 2$, Wu [22] studied the equivalence of the radial exponent of convergence of zeros and the radial order of the product of a solution base for the homogeneous linear differential equation

$$f^{(k)} + A_{k-2} f^{(k-2)} + \cdots + A_0 f = 0, \tag{2}$$

where $A_0, \cdots, A_{k-2}$ are entire functions of infinite order with $A_0 \not\equiv 0$. In this paper, we continue to study the connection of the radial exponent of convergence of zeros with the radial order of the product of a solution base for linear differential equation (2) with entire coefficients of finite iterated order. For the convenience, we define inductively (see [3, 21]), for $r \in [0, +\infty)$, $\exp^{[1]} r = e^r$ and $\exp^{[n+1]} r = \exp(\exp^{[n]} r), n \in \mathbb{N}$. For all r sufficiently large, we define $\log^{[1]} r = \log r$ and $\log^{[n+1]} r = \log(\log^{[n]} r), n \in \mathbb{N}$. We also denote $\exp^{[0]} r = r = \log^{[0]} r$, $\log^{[-1]} r = \exp^{[1]} r$ and $\exp^{[-1]} r = \log^{[1]} r$. We recall the following definitions and remarks.

Definition 3 [11, 15] The iterated p-order $\sigma_p(f)$ of a meromorphic function $f(z)$ is defined by

$$\sigma_p(f) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log r} \quad (p \in \mathbb{N}).$$

Remark 1 [4] 1) If $p=1$, then we denote $\sigma_1(f) = \sigma(f)$; 2) If $p=2$. then we denote the so-called hyper order by $\sigma_2(f)$; 3) If $f(z)$ is an entire function, then

$$\sigma_p(f) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log r}.$$