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QUASI-SURE CONVERGENCE RATE OF EULER SCHEME FOR STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract Let $X_t(x)$ be the solution of stochastic differential equations with smooth and bounded derivatives coefficients. Let $X^n_t(x)$ be the Euler discretization scheme of SDEs with step 2^{-n}. In this note, we prove that for any $R>0$ and $\gamma\in(0,1/2)$,

$$\sup_{t\in[0,1],|x|\leqslant R}|X^n_t(x,\omega)-X_t(x,\omega)|\leqslant \xi_{R,\gamma}(\omega)2^{-n\gamma},\quad n\geqslant 1,\quad \text{q.e.},$$

where $\xi_{R,\gamma}(\omega)$ is quasi-everywhere finite.

Key words Euler approximation; quasi-sure convergence; SDE
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1 Introduction

Consider the following stochastic differential equation (SDE) of Itô's type :

$$\begin{cases} \mathrm{d}X_t=\sigma(X_t)\cdot \mathrm{d}W_t+b(X_t)\mathrm{d}t,\\ X_0=x\in\mathbb{R}^d, \end{cases} \tag{1.1}$$

where $\{W_t\}_{t\in[0,1]}$ is an m-dimensional standard Brownian motion defined on the classical Wiener space $(\Omega,\mathcal{F},\mathcal{P})$, $\{\sigma^i_j, i=1,\cdots,d, j=1,\cdots,m\}$ and $\{b^i, i=1,\cdots,d\}$ are bounded smooth functions on $\mathbb{R}^d$ with bounded derivatives of all orders. The unique solution is denoted by $X_t(x)$.

The Euler scheme of SDE (1.1) is defined by

$$X^n_t=x+\int_0^t\sigma(X^n_{s_n})\cdot \mathrm{d}W_s+\int_0^t b(X^n_{s_n})\mathrm{d}s,\quad n\in\mathbb{N},$$

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or recursively,

$$X_t^n = X_{t_n}^n + \sigma(X_{t_n}^n)(W_t - W_{t_n}) + b(X_{t_n}^n)(t - t_n), \tag{1.2}$$

where $s_n := \frac{[2^n s]}{2^n}$, and $[a]$ denotes the integer part of real number a.

Up to now, there were many papers devoted to the study of the various convergence for Euler's scheme (see [2, 3, 6, 14] etc.). It is interesting that the Euler approximation even can be used to construct the solutions for SDEs with discontinuous coefficients in Gyöngy-Krylov [3]. On the other hand, for the aim of numerical calculations, in [1], Bally and Talay studied the convergence rate of the distribution function for Euler scheme.

As we known, in the classical probability theory, one can ignore a null set in the sense of probability measure. However, a more delicate analysis in potential theory shows that the zero probability set can not always be ignored. Since Malliavin [7] created the stochastic calculus of variation in 1976, the analysis over infinite dimensional spaces was developed extensively. Meanwhile, in the paper [8], Malliavin also initiated the quasi-sure analysis, which is finer than almost-sure analysis in probability theory. More introduction about the quasi-sure analysis can be found in Malliavin [9], Ren [13] and Huang-Yan [4].

Basing on this consideration, in this paper we mainly prove that

Theorem 1.1 For any $R > 0$ and $\gamma \in (0, \frac{1}{2})$, there are slim set N and a quasi everywhere finite random variable $\xi_{R,\gamma}$ such that for all $\omega \in N^c$,

$$\sup_{t\in[0,1],|x|\leqslant R} |X_t^n(x,\omega) - X_t(x,\omega)| \leqslant \xi_{R,\gamma}(\omega)2^{-n\gamma}, \quad n \geqslant 1.$$

The proof of this theorem is based on a Doob's inequality in terms of (p,k)-capacity established in Ren [12] together with some necessary estimates. After some preliminaries in Section 2, we shall prove this result in Section 3. Throughout the paper, C with or without indexes will denote different constants, whose values are not important.

2 Preliminaries

We will work on the canonical probability space $(\Omega, \mathcal{F}, \mathcal{P}; \mathbb{H})$, where Ω is the space of continuous functions on $[0,1]$ starting at zero, and endowed with the topology of the uniform convergence, P the standard Wiener measure, $\mathcal{F}$ the completion of the Borel σ-field of Ω with respect to P, $\mathbb{H}$ the Cameron-Martin subspace, i.e., it consists of functions $h : [0,1] \to \mathbb{R}$ which are absolutely continuous and whose derivative $\dot{h}$ belongs to $L^2([0,1])$; $\mathbb{H}$ is then a Hilbert space with the inner product

$$\langle h_1, h_2\rangle_{\mathbb{H}} = \int_0^1 \dot{h}_1(t)\dot{h}_2(t)\mathrm{d}t.$$

For $h \in \mathbb{H}$, let $W(h) := \int_0^1 \dot{h}(t)\mathrm{d}W_t$. Let us first recall some elementary facts about the Malliavin calculus (cf. [4, 9]). Let $\mathcal{C}$ be the smooth functional space defined as follows,

$$\mathcal{C} := \Big\{F = f(W(h_1), \cdots, W(h_n)) : f \in C_b^\infty(\mathbb{R}^n, \mathbb{R}), h_i \in \mathbb{H}, 1 \leqslant i \leqslant n; n \in \mathbb{N}\Big\}.$$

For $F \in \mathcal{C}$ and $h \in \mathbb{H}$, one defines the gradient operator

$$\langle DF, h\rangle_{\mathbb{H}} = \sum_{i=1}^n \partial_i f(W(h_1), \cdots, W(h_n))\langle h_i, h\rangle_{\mathbb{H}}.$$