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ON INTERSECTIONS OF INDEPENDENT NONDEGENERATE DIFFUSION PROCESSES*

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Abstract Let $X^{(1)} = \{X^{(1)}(s), s \in \mathbb{R}_+\}$ and $X^{(2)} = \{X^{(2)}(t), t \in \mathbb{R}_+\}$ be two independent nondegenerate diffusion processes with values in $\mathbb{R}^d$. The existence and fractal dimension of intersections of the sample paths of $X^{(1)}$ and $X^{(2)}$ are studied. More generally, let $E_1, E_2 \subseteq (0,\infty)$ and $F \subset \mathbb{R}^d$ be Borel sets. A necessary condition and a sufficient condition for $\mathbb{P}\{X^{(1)}(E_1) \cap X^{(2)}(E_2) \cap F \neq \varnothing\} > 0$ are proved in terms of the Bessel-Riesz type capacity and Hausdorff measure of $E_1 \times E_2 \times F$ in the metric space $(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d, \widehat{\rho})$, where $\widehat{\rho}$ is an unsymmetric metric defined in $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d$. Under reasonable conditions, results resembling those of Browian motion are obtained.

Key words intersection; diffusion processes; hitting probability; polar set; Hausdorff dimension

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1 Introduction

Intersections of the sample paths of stochastic processes were studied by many authors. It is well known that the trajectories of two independent Brownian motions can inersect if and only if the spatial dimention d is at most three. See Dvoretzky et al [1] and Khoshnevisan [2]. These results on intersections of Brownian motion were extended in various ways to Lévy processes, Gaussian processes and other processes. We refer to the survey papers of Taylor [3] and Xiao [4] for further information. Recently, Chen and Xiao [5] studied intersections of two independent anisotropic Gaussian fields.

This paper is concerned with the existence and fractal dimension of intersections of nondegenerate diffusion processes with values in $\mathbb{R}^d$. The approach in Chen and Xiao [5] is based on the results on hitting probabilities of Gaussian random fields in Biermé, Lacaux and Xiao [6] and Xiao [7]. This is different from the approaches based on intersection local times or Gaussian property in Rosen [8, 9], Hu and Nualart [10], Wu and Xiao [11]. In this paper, we

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will extend the approach in Chen and Xiao [5] to study intersections of nondegenerate diffusion processes, which are non-Gaussian stochastic processes. Our arguments are based on the results on hitting probabilities of nondegenerate diffusion processes in Section 2.

Let $B = \{B(t), t \geq 0\}$ be the d-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by $B(t) = (B_1(t), B_2(t), \cdots, B_d(t))$. For all $1 \leq i \leq d$, let $\beta_i : \mathbb{R}^d \to \mathbb{R}$ be measurable functions, and let $\alpha(x) = (\alpha_{ij}(x),\ 1 \leq i,\ j \leq d)$ be a measurable $d \times d$ function matrix defined in $\mathbb{R}^d$. Throughout this paper we assume that $\alpha(x)$ and $\beta(x)$ are bounded and continuous functions. For all $x \in \mathbb{R}^d$, let $a(x) = (a_{ij}(x),\ 1 \leq i,\ j \leq d)$, where $a_{ij}(x) = \sum\limits_{k=1}^{d} \alpha_{ik}(x)\alpha_{kj}(x)$. We assume that $\alpha(x)$ and $\beta(x)$ satisfy the following conditions.

C1 There exists a positive and finite constant c such that for all $x = (x_1, x_2, \cdots, x_d)$, $y = (y_1, y_2, \cdots, y_d) \in \mathbb{R}^d$, we have

$$\begin{aligned} & |\alpha(x) - \alpha(y)| + |\beta(x) - \beta(y)| \\ = & \bigg(\sum_{i=1}^{d} \sum_{j=1}^{d} (\alpha_{ij}(x) - \alpha_{ij}(y))^2 \bigg)^{\frac{1}{2}} + \bigg(\sum_{i=1}^{d} (\beta_i(x) - \beta_i(y))^2 \bigg)^{\frac{1}{2}} \\ \leq & \, c|x - y| = c\bigg(\sum_{i=1}^{d} (x_i - y_i)^2 \bigg)^{\frac{1}{2}}. \end{aligned}$$

C2 There exists a positive and finite constant λ such that for all $x \in \mathbb{R}^d$ and for all $e \in \mathbb{R}^d$ with $|e| = 1$, we have $e'a(x)e \geq \lambda$. That is, the eigenvalues of $a(x)$ are bounded from below.

Consider the following stochastic differential equation introduced by Sheu [12]:

$$X(t) = x_0 + \int_0^t \alpha(X(s)) \mathrm{d}B(s) + \int_0^t \beta(X(s)) \mathrm{d}s,\ t \geq 0, \tag{1.1}$$

where $x_0 = (x_{01}, x_{02}, \cdots, x_{0d})$ is a given constant in $\mathbb{R}^d$ and $X(t) = (X_1(t), X_2(t), \cdots, X_d(t))$. If the solution of (1.1) exists, we call each solution $\{X(t), t \geq 0\}$ of (1.1) a nondegenerate diffusion process with values in $\mathbb{R}^d$. The properties of nondegenerate diffusion processes were studied by many authors. See Ikeda [13], Sheu [12], Yang [14–16] and Chen [17] for further information.

Let $B^{(1)} = \{B^{(1)}(s), s \geq 0\}$ and $B^{(2)} = \{B^{(2)}(t), t \geq 0\}$ be two independent Brownian motions taking values in $\mathbb{R}^d$. For any $k \in \{1, 2\}$, we assume that $\alpha^{(k)}(x)$ and $\beta^{(k)}(x)$ satisfy conditions C1 and C2. Then, the stochastic differential equation (1.1) has a unique strong solution $X^{(k)} = \{X^{(k)}(t), t \geq 0\}$ (cf. Ikeda et al. [13]). It is easy to see that $X^{(1)} = \{X^{(1)}(s), s \geq 0\}$ and $X^{(2)} = \{X^{(2)}(t), t \geq 0\}$ are two independent nondegenerate diffusion processes taking values in $\mathbb{R}^d$.

We say the nondegenerate diffusion processes $X^{(1)}$ and $X^{(2)}$ intersect if there exist $s \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$ such that $X^{(1)}(s) = X^{(2)}(t)$. In this paper we study the existence of intersections and, particularly, the following problems.

(i) When do $X^{(1)}$ and $X^{(2)}$ intersect (with positive probability)?

(ii) Let $E_1 \subseteq (0, \infty)$ and $E_2 \subseteq (0, \infty)$ be arbitrary Borel sets. When do $X^{(1)}$ and $X^{(2)}$ intersect if we restrict the "time" $s \in E_1$ and $t \in E_2$? More precisely, when is

$$\mathbb{P}\Big\{ X^{(1)}(E_1) \cap X^{(2)}(E_2) \neq \varnothing \Big\} > 0?$$