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HOMOGENIZATION OF THE INCOMPRESSIBLE NAVIER-STOKES FLUID WITH OSCILLATION COEFFICIENT*

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Abstract We study the homogenization of the incompressible Navier-Stokes equations with periodic oscillating coefficient in a bounded non-homogeneous media. To do that, we introduce a generalized compensate compactness result and a suitable class of test function to this problem. By passing the limit, we obtain the homogenized model of this problem.

Key words Div-curl lemma; homogenization; Navier-Stokes equations

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1 Introduction

In mathematical models of microscopically non-homogeneous media, various local characteristics are usually described by functions of $c(\frac{x}{\varepsilon})$, where $\varepsilon > 0$ is a small parameter. The function $c(x)$ can be periodic or belongs to some other specific class. It is an extremely difficult problem to study the properties of the micro non-homogenous medium. A possible way is to attack the problem by applying asymptotic analysis, which immediately leads to the concept of homogenization. To characterize this homogenous medium is one of the main tasks in the homogenization theory. For more information concerning the homogenization theory, the readers are referred to [1, 2], etc.

There are many literatures on the homogenization of microscopically non-homogeneous media. Peter Wall [3] studied the homogenization and corrector results for some nonlinear monotone operators. Johan Bystöm [4] discussed the same problem in a different Sobolev space. By using Bloch decomposition, Allaire [5–7] considered the homogenization of the Schrödinger equation, hyperbolic equation, parabolic equation and elliptic equation in a locally periodic medium. To solve the homogenization of different models of microscopically non-homogeneous media, Bensoussan [1] and Sanchez-Palencia [2] introduced the formal extension method with different scales. They also gave a rigorous proof of those homogenized models. Courilleau [8]

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studied the homogenization of the elliptic equation by compensated compactness method in a bounded domain which has a thin structure. Other works can be seen in [9, 10, 17, 18] and the references therein.

There are a few works on the homogenization of viscosity Navier-Stokes fluid, where the fluid's viscosity depends on the spatial variables. We also assume that the viscosity has a rapid oscillation. The initial motivation of this study is the limit behavior, when ε takes a sequence of positive values tending to zero, of the incompressible Navier-Stokes fluid with oscillating coefficient

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + (u_\varepsilon \cdot \nabla)u_\varepsilon - \operatorname{div}(\mathcal{A}(\dfrac{x}{\varepsilon})\nabla u_\varepsilon) + \nabla p_\varepsilon = f, & \text{in } \Omega \times (0,T), \\ \operatorname{div} u_\varepsilon = 0, & \text{in } \Omega \times (0,T), \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N with smooth boundary and T is a finite positive number. $u_\varepsilon,\ p_\varepsilon$ denote the velocity and the pressure of the fluid respectively. $\mathcal{A}(\frac{x}{\varepsilon})$ is a periodic, symmetric and positive-define matrix. In this paper, we assume that for any $n \times n$ matrix ξ, there exist positive constants λ, Λ such that

$$\lambda|\xi|^2 \le \mathcal{A}(\frac{x}{\varepsilon})\xi : \xi \le \Lambda|\xi|^2.$$

The system is complement with no-slip boundary condition

$$u_\varepsilon = 0 \qquad \text{on } \partial\Omega \text{ for } t \in (0,T). \tag{1.2}$$

We consider the system under the following initial condition

$$u_\varepsilon(x,0) = u_{\varepsilon,0}(x) \text{ in } \Omega. \tag{1.3}$$

Our goal is to find the asymptotic behavior of $u_\varepsilon,\ p_\varepsilon$ as ε tends to zero. The main stumbling block here is the fact that the product of two weak sequences maybe doesn't converge to the product of their weak limits. To be precise, based on the physical view, we can only assume the weak convergence of $\mathcal{A}(\frac{x}{\varepsilon})$, we should deal with the limits of the inertia term and the viscous force term. In order to overcome those difficulties, we generalize the classical compensate compactness result (see [11, 12]) and use the trick of construction the test function.

The advantage of considering the homogenized problem instead of the original one is obvious, especially in numerical computing. It will be hard to deal with (1.1) because of the oscillating coefficient. As we will see later, the coefficient in the homogenized model is a constant. It is much easier both in theory analysis and numerical computing.

2 Preliminaries and the Main Result

Let Ω be an open bounded subset of R^n, $n = 2$ or 3. It is also supposed to be local periodic with periodicity Y. Let ε be a subsequence of real positive numbers such that $\varepsilon \to 0^+$. $W^{1,p}(\Omega)(= H^1(\Omega)$ if $p = 2)$ is the standard Sobolev space which contains the functions whose derivatives of order one and itself belong to $L^p(\Omega)$. $W^{-1,p'}(\Omega)$ is the dual of $W^{1,p}(\Omega)$ and $\frac{1}{p} + \frac{1}{p'} = 1$. $W_0^{1,p}(\Omega)$ is the subset of $W^{1,p}(\Omega)$ with trace 0 on $\partial\Omega$. $Q_T = \Omega \times (0,T)$. Throughout this paper, C will be a constant that may differ from one place to another. v' means the derivative with respect to t.