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LOCAL WELL-POSEDNESS IN SOBOLEV SPACES WITH NEGATIVE INDICES FOR A SEVENTH ORDER DISPERSIVE EQUATION*

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Abstract This paper is concerned with the Cauchy problem of a seventh order dispersive equation. We prove local well-posedness with initial data in Sobolev spaces $H^s(R)$ for negative indices of $s > -\frac{11}{4}$.

Key words Cauchy problem; local well-posedness; Sobolev spaces; bilinear estimate

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1 Introduction

In this work, we consider the Cauchy problem for the seventh order dispersive equation

$$\partial_t u + \alpha\partial_x^7 u + \beta\partial_x^5 u + \gamma\partial_x^3 u + u\partial_x u = 0, \tag{1.1}$$

$$u(x,0) = \varphi(x), \tag{1.2}$$

where $\alpha \neq 0$, β, γ are real constants and the initial data $\varphi \in H^s(R)$.

Equations of this type arise as mathematical models for the weakly nonlinear propagation of long waves [10]. Note that the case $\alpha = 0, \beta \neq 0$ corresponds to the Kawahara equation and the case $\alpha = 0, \beta = 0, \gamma = 1$ corresponds to the KdV equation. The local well-posedness for dispersive equations with quadratic nonlinearities was extensively studied in Sobolev spaces with negative indices. The proof of these results were based on the Fourier restriction norm approach introduced by Bourgain in his study of the nonlinear Schrödinger equation [14] and the KdV equation [15]. This method was further developed by Kenig, Ponce and Vega [11]. They proved local well-posedness for the KdV equation with the initial data in Sobolev space $H^s(R)$ with negative indices $-\frac{3}{4} < s \leqslant 0$. Kenig, Ponce and Vega [12] also studied the following high-order dispersive equation

$$u_t + \partial_x^{2j+1} u + P(u, \partial_x, \cdots, \partial_x^{2j} u) = 0$$

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and obtained local well-posedness for initial data $u_0 \in H^s(R) \cap L^2(|x|^m \mathrm{d}x)$, $s, m \in Z^+$, where P is a polynomial without constant or linear terms. As for Kawahara equation, [13] proved local well posedness in $H^s(R)$ for $s > -\frac{7}{4}$ by means of $[k; Z]$-multiplier norm method [1].

The purpose of this paper is to obtain local well-posedness for (1.1), (1.2) with low regularity data. To this end, we first derive a fundamental estimate on dyadic blocks for the seventh order dispersive equation. We then apply this fundamental estimate to establish new bilinear estimate in Bourgain spaces. Combining these estimates with a contraction mapping argument, we can establish the well-posedness theorem.

Next we give the precise definition of the $X_{s,b}$ spaces related to our problem.

Definition 1.1 For $s, b \in R$, let $X_{s,b}$ denotes the completion of the Schwartz functions with respect to the norm

$$\|u\|_{X_{s,b}} = \|\langle \tau - \phi(\xi) \rangle^b \langle \xi \rangle^s \widehat{u}(\xi, \tau)\|_{L^2_{\xi,\tau}},$$

where $\phi(\xi) = \alpha\xi^7 - \beta\xi^5 + \gamma\xi^3$ and "$\wedge$" denotes the time space Fourier transform, and $\langle a \rangle = \sqrt{1 + |a|^2}$.

We will also need the localized $X_{s,b}$ defined as follows.

Definition 1.2 For $s, b \in R$ and $\delta \geqslant 0$, we have that $X^\delta_{s,b}$ denotes the space endowed with the norm

$$\|u\|_{X^\delta_{s,b}} = \inf_{w \in X_{s,b}} \{\|w\|_{X_{s,b}} : w(t) = u(t) \ \text{ on } \ [0, \delta]\}.$$

Now we state the main results of this paper.

Theorem 1.1 For $s > -\frac{11}{4}$ and $u, v \in X_{s,b}$, there exists $b \in (\frac{1}{2}, 1)$ such that the bilinear inequality

$$\|\partial_x(uv)\|_{X_{s,b-1}} \leqslant C\|u\|_{X_{s,b}}\|v\|_{X_{s,b}} \tag{1.3}$$

holds with a constant $C > 0$ depending only on s and b.

Theorem 1.2 Let $s > -\frac{11}{4}$, then for all $\varphi \in H^s(R)$, there exist $T = T(\|\varphi\|_{H^s(R)})$ and a unique solution u of the initial value problem associated to equation (1.1) with initial data $u(0) = \varphi$ such that

$$u \in C([0, T]; H^s(R)) \cap X^T_{s,b}.$$

Moreover, given $T' \in (0, T)$ there exists $M = M(T') > 0$ such that given the set

$$W = \{(\tilde{\varphi}, \tilde{\psi}) \in H^s(R) \times H^{s-1}(R) : \|\tilde{\varphi} - \varphi\|^2_{H^s(R)} + \|\tilde{\psi} - \psi\|^2_{H^{s-1}(R)} < M\},$$

the map solution

$$S : W \to C([0, T']; H^s(R)) \cap X^T_{s,b}, \quad (\tilde{\varphi}, \tilde{\psi}) \to u(t)$$

is Lipschitz.

The plan of this paper is as follows. In Section 2, we prove a fundamental estimates on dyadic blocks for the seventh order dispersive equation. Bilinear estimate is proved in Section 3. Finally, the Cauchy problem is treated in Section 4.