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CRITICAL EXPONENTS AND CRITICAL DIMENSIONS FOR NONLINEAR ELLIPTIC PROBLEMS WITH SINGULAR COEFFICIENTS*

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Abstract Let $B_1 \subset \mathbb{R}^N$ be a unit ball centered at the origin. The main purpose of this paper is to discuss the critical dimension phenomenon for radial solutions of the following quasilinear elliptic problem involving critical Sobolev exponent and singular coefficients:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = |x|^s|u|^{p^*(s)-2}u + \lambda|x|^t|u|^{p-2}u, \ x \in B_1, \\ u|_{\partial B_1} = 0, \end{cases}$$

where $t, s > -p, 2 \le p < N, p^*(s) = \frac{(N+s)p}{N-p}$ and λ is a real parameter. We show particularly that the above problem exists infinitely many radial solutions if the space dimension $N > p(p-1)t + p(p^2 - p + 1)$ and $\lambda \in (0, \lambda_{1,t})$, where $\lambda_{1,t}$ is the first eigenvalue of $-\Delta_p$ with the Dirichlet boundary condition. Meanwhile, the nonexistence of sign-changing radial solutions is proved if the space dimension $N \le \frac{(ps+p)\min\{1,\frac{p+t}{p+s}\}+p^2}{p-(p-1)\min\{1,\frac{p+t}{p+s}\}}$ and $\lambda > 0$ is small.

Key words singular coefficients; radial solution; critical exponent; p-Laplace equations

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1 Introduction and Main Results

It is well known from the work of Brézis and Nirenberg [1] that the existence of positive solutions of semilinear elliptic equations involving critical exponents relate to the dimension of space. More specially, for the representative problem

$$\begin{cases} -\Delta u = \lambda u + u^{2^*-1}, \ x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded smooth open subset of $\mathbb{R}^N, N \geq 3$ and $2^* = \frac{2N}{N-2}$ is the critical exponent for Sobolev embedding. The following results were proved in [1]:

(i) If $N \geq 4$, problem (1.1) has at least one solution $u \in H_0^1(\Omega)$ when $0 < \lambda < \lambda_1$;

(ii) If $N = 3$, problem (1.1) has at least one solution $u \in H_0^1(\Omega)$ when $\lambda_* < \lambda < \lambda_1$, where λ_* is a positive constant;

(iii) If $N = 3$ and Ω is a ball, then $\lambda_* = \frac{1}{4}\lambda_1$, and problem (1.1) has no solution for $\lambda \leq \lambda_*$, where λ_1 is the first eigenvalue of the operator $-\Delta$ in Ω with Dirichlet boundary condition.

The preceding results show that the space dimension N plays a fundamental role when people seeks solutions of (1.1). In particular, the dimension $N = 3$ is a special one if compared with $N \geq 4$. According to the definition introduced by Pucci and Serrin (see [2], also see [3]), we shall say that $N = 3$ is a critical dimension for problem (1.1). In the celebrated papers [2] and [3], a wide class of nonlinear critical elliptic problems which exhibit the phenomenon of critical dimensions have been studied.

The main purpose of this paper is to show that the same critical dimension phenomenon also appears in the study of the sign-changing solutions for the quasilinear elliptic problems with singular coefficients, for example

$$\begin{cases} -\Delta_p u = \lambda |x|^t |u|^{p-2}u + |x|^s |u|^{p^*(s)-2}u, \ x \in B_1, \\ u\big|_{\partial B_1} = 0, \end{cases} \tag{1.2}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \ t, s > -p, \ 2 \leq p < N, \ p^*(s) = \frac{(N+s)p}{N-p}$, λ is a real parameter, $B_1 \subset \mathbb{R}^N$ is a unit ball centered at the origin, and $p^*(s)$ is the critical Sobolev–Hardy exponent with the embedding $W_0^{1,p}(B_1) \hookrightarrow L^{p^*(s)}(|x|^s, B_1)$.

A dimension N is called a critical dimension for sign-changing solutions of (1.2) if problem (1.2) has no sign-changing radial solutions for some $\lambda > 0$.

There have been so many works exhibited the phenomenon of critical dimension for the sign-changing radial solutions of (1.2) when $s = t = 0$. For the known results and related topics including the existence and nonexistence of nodal solutions, we refer the readers to [4–15]. Especially when $0 < \lambda < \lambda_1$, the following problem was considered:

$$\begin{cases} -\Delta u = \lambda |u|^{q-2}u + |u|^{2^*-2}u, \ x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{in} \quad \Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded smooth open subset of $\mathbb{R}^N, \ N \geq 3, \ 2 \leq q < 2^*$ and $2^* = \frac{2N}{N-2}$ is the critical exponent for Sobolev embedding. Cerami et al. [16] and Chen [17] proved that problem (1.3) admits a pair of nodal solutions with exactly k nodes for any integer $k > 0$ if $N \geq 7$, $q = 2$ and $\Omega = B_1$. Tarantello [18] obtained the existence of nodal solutions of (1.3) by the convergence of nodal solutions of subcritical problems with $2^* - \epsilon$ as $\epsilon \to 0$. When $\Omega = B_1$ and $\max\{2, \ \frac{N+2}{N-2}\} < q < \frac{2N}{N-2}$, the existence of infinitely many nodal radial solutions for problem (1.3) was obtained in [3]. Adimurthi–Yadava [5] and Wang–Wu [9] considered the nonexistence of nodal radial solutions for (1.5) when $q = 2, \ N \in \{3, \ 4, \ 5, \ 6\}$ by using Pohozăev's identity.

Recently, we extended problem (1.3) to more general case

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + |u|^{p^*-2}u, \ x \in B_1, \\ u|_{\partial B_1} = 0, \quad u > 0 \quad \text{in} \quad B_1, \end{cases} \tag{1.4}$$