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BASISITY PROBLEM AND WEIGHTED SHIFT OPERATORS*

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Abstract We investigate a basisity problem in the space $\ell_A^p(\mathbb{D})$ and in its invariant subspaces. Namely, let W denote a unilateral weighted shift operator acting in the space $\ell_A^p(\mathbb{D})$, $1 \le p < \infty$, by $Wz^n = \lambda_n z^{n+1}, n \ge 0$, with respect to the standard basis $\{z^n\}_{n\ge 0}$. Applying the so-called "discrete Duhamel product" technique, it is proven that for any integer $k \ge 1$ the sequence $\{(w_{i+nk})^{-1}(W \mid E_i)^{kn} f\}_{n\ge 0}$ is a basic sequence in $E_i := \operatorname{span}\{z^{i+n} : n \ge 0\}$ equivalent to the basis $\{z^{i+n}\}_{n\ge 0}$ if and only if $\widehat{f}(i) \ne 0$. We also investigate a Banach algebra structure for the subspaces $E_i, i \ge 0$.

Key words basis; basic sequence; discrete Duhamel product; Banach algebra; weighted shift operator

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1 Introduction and Preliminaries

In [4], the author investigated basisity problem in the space $\ell_A^p := \ell_A^p(\mathbb{D})$, $1 \le p < \infty$, consisting from the analytic functions $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n) z^n$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for which $\|f\|_p^p := \sum\limits_{n=0}^{\infty} |\widehat{f}(n)|^p < +\infty$, where $\widehat{f}(n) := \frac{f^{(n)}(0)}{n!}$ is the n th Taylor coefficient of the function f. Namely, it is characterized in [4, Theorem 2.1] those function $f(z)$ in $\ell_A^p(\mathbb{D})$ for which the sequence $\{(nk)!V^{nk}f\}_{n=0}^{\infty}$ in $\ell_A^p(\mathbb{D})$ is a basic sequence equivalent to the standard basis $\{z^n\}_{n=0}^{\infty}$ of $\ell_A^p(\mathbb{D})$; here V is the Volterra integration on $\ell_A^p(\mathbb{D})$ defined by $Vf(z) = \int_0^z f(t)\mathrm{d}t$.

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In the present article, in particular, we generalize this result for the weighted shift operator W defined on $\ell_A^p(\mathbb{D})$ by the formula $Wz^n = \lambda_n z^{n+1}$, where $\lambda_n \neq 0$, $n \geq 0$, are complex numbers with $\sup\limits_{n\geq 0} |\lambda_n| < +\infty$ (see Corollary 2.1 below).

For more informations about bases in spaces of holomorphic functions the readers can be found, for instance, in Haslinger [6], Holub [7] and Fage and Nagnibida [3].

Before giving our results, let us introduce some necessary notations and preliminaries.

Recall that a sequence $(x_n)_{n\geq 0}$ in a Banach space X is said to be a basis for X if every vector $x \in X$ can be written uniquely as a convergent series $x = \sum\limits_{n=0}^{\infty} a_n x_n$. The sequence $(x_n)_{n\geq 0}$ is called a basic sequence (see, for instance, Holub [7], Singer [16] and Gürdal [4]) if it forms a basis for its closed linear span in X. If $(x_n)_{n\geq 0}$ is a basis for X and $(y_n)_{n\geq 0}$ a basis for Y, we say $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ are equivalent if the operator $A : X \to Y$ defined by $Ax_n = y_n$, $n \geq 0$, is a linear homeomorphism, i.e., if $\sum\limits_{n=0}^{\infty} a_n x_n$ converges in X if and only if $\sum\limits_{n=0}^{\infty} a_n y_n$ converges in Y [16, p.70].

We put $E_i := \operatorname{span}\{z^k : k \geq i\}$, $i = 1, 2, \cdots$, and $w_n := \lambda_0 \lambda_1 \cdots \lambda_{n-1}$, $w_0 := 1$. Clearly, $E_0 = \ell_A^p(\mathbb{D})$ and $WE_i \subset E_i$ for each $i \geq 0$, that is, E_i is a W-invariant subspace.

Given two functions $f(z) = \sum\limits_{n=i}^{\infty} \widehat{f}(n) z^n$ and $g(z) = \sum\limits_{n=i}^{\infty} \widehat{g}(n) z^n$ in E_i $(i \geq 0)$, their so-called discrete Duhamel product $\underset{i}{\circledast}$ (see [8]) is defined by

$$\big(f \underset{i}{\circledast} g\big)(z) := \sum_{n=i}^{\infty} \sum_{m=i}^{\infty} \frac{w_{n+m-i}}{w_n w_m} \widehat{f}(n) \widehat{g}(m) z^{n+m-i}. \tag{1}$$

It is easy to see from (1) that the classical Duhamel product

$$\big(f \circledast g\big)(z) := \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z f(z-t) g(t) \mathrm{d}t$$

corresponds to the case $\lambda_n = \frac{1}{n+1}$, $n \geq 0$, and $i = 0$.

Our approach in this article is based on some properties of the discrete Duhamel product $\underset{i}{\circledast}$ in the subspace E_i. The following lemma can be proved by using the same arguments as in [8, Proof of Theorem 4] (and therefore we omit its proof); see also in [9, 10].

Lemma 1.1 Let $(\lambda_n)_{n\geq 0}$ be a bounded sequence of complex numbers such that

$$\sum_{n\geq N} \sum_{m\geq N} \left| \frac{w_{n+m-i}}{w_n w_m} \right|^q < \infty$$

for some integer $N = N_i \geq i$. Let $p \in (1, \infty)$ and q be the conjugate exponent to p (i.e., $\frac{1}{p} + \frac{1}{q} = 1$). Then the following assertions are true:

(i) There exists a constant $C_i > 0$ such that

$$\Big\| f \underset{i}{\circledast} g \Big\|_{E_i} \leq C_i \, \|f\|_{E_i} \, \|g\|_{E_i} \tag{2}$$

for all $f, g \in E_i$, i.e., the subspace $E_i \subset \ell_A^p(\mathbb{D})$ is a unitial Banach algebra with respect to the discrete Duhamel product $\underset{i}{\circledast}$ with the unit $w_i z^i$.

(ii) An element $f \in E_i$ is $\underset{i}{\circledast}$-invertible if and only if $\widehat{f}(i) \neq 0$.