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VARIOUS NOTIONS OF ORTHOGONALITY IN NORMED SPACES*

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Abstract In this paper, we present various notions and aspects of orthogonality in normed spaces. Characterizations and generalizations of orthogonality are also considered. Results on orthogonality of the range and the kernel of elementary operators and the operators implementing them also are given.

Key words orthogonality; normed space; elementary operator

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1 Introduction

Orthogonality in normed spaces is a concept that was analyzed for quite a period of time. Benitez [4] described several types of orthogonality which were studied in real normed spaces namely: Robert's orthogonality, Birkhoff's orthogonality, Orthogonality in the sense of James, Isoceles, Pythagoras, Carlsson, Diminnie, Area among others. Some of these orthogonalities are described as follows. For $x \in \mathcal{M}$ and $y \in \mathcal{N}$ where $\mathcal{M}$ and $\mathcal{N}$ are subspaces of E which is a normed linear space, we have

(i) Roberts: $\|x - \lambda y\| = \|x + \lambda y\|,\ \forall \lambda \in \mathbb{R}$.

(ii) Birkhoff: $\|x + y\| \geq \|y\|$.

(iii) Isosceles: $\|x - y\| = \|x + y\|$.

(iv) Pythagorean: $\|x - y\|^2 = \|x\|^2 + \|y\|^2$.

(v) a-Pythagorean: $\|x - ay\|^2 = \|x\|^2 + a^2\|y\|^2,\ a \neq 0$.

(vi) Diminnie: $\sup\{f(x)g(y) - f(y)g(x) : f,\ g \in S'\} = \|x\|\|y\|$ where S' denotes the unit sphere of the topological dual of E.

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(vii) Area: $\|x\|\|y\| = 0$ or they are linearly independent and such that $x, -x, y, -y$ divide the unit ball of their own plane (identified by $\mathbb{R}^2$) in four equal areas.

Consider a normed space $\mathcal{A}$ and let $T_{A,B} : \mathcal{A} \to \mathcal{A}$. T is called an elementary operator if it has the following representation:

$$T(X) = \sum_{i=1}^{n} A_i X B_i, \ \forall\, X \in \mathcal{A},$$

where $A_i,\ B_i$ are fixed in $\mathcal{A}$. Let $\mathcal{A} = B(H)$. For $A,\, B \in B(H)$ we define the particular elementary operators:

(i) The left multiplication operator $L_A : B(H) \to B(H)$ by $L_A(X) = AX,\ \forall\, X \in B(H)$.

(ii) The right multiplication operator $R_B : B(H) \to B(H)$ by $R_B(X) = XB,\ \forall\, X \in B(H)$.

(iii) The generalized derivation (implemented by $A,\ B$) by $\delta_{A,B} = L_A - R_B$.

(iv) The basic elementary operator(implemented by $A,\ B$) by $M_{A,\,B}(X) = AXB,\ \forall\, X \in B(H)$.

(v) The Jordan elementary operator(implemented by $A,\ B$) by $\mathcal{U}_{A,\,B}(X) = AXB + BXA,\ \forall\, X \in B(H)$.

Regarding orthogonality involving elementary operators, Anderson [1] established the orthogonality of the range and kernel of normal derivations. Others who also worked on orthogonality include: Kittaneh [22], Mecheri [30] among others. For details see [1–28, 32–34]. We shall investigate the orthogonality of the range and the kernel of several types of important elementary operators.

Anderson [1] in his investigations proved that if N and S are operators in $B(H)$ such that N is normal and $NS = SN$ then for all $X \in B(H)$, $\|\delta_N(X) + S\| \geq \|S\|$. If S(above) is a Hilbert-Schmidt operator then Kittaneh [22] (see also the references therein) showed that $\|\delta_N(X) + S\|_2^2 = \|\delta_N(X)\|_2^2 + \|S\|_2^2$. We extend this study to the $\mathcal{NA}(\mathcal{H})$ and $\mathcal{NA}(\mathcal{H})$-classes.

2 Orthogonality in $\mathcal{NA}(\mathcal{H})$ and $\mathcal{NA}(\mathcal{H})$-Classes

First we note that $\mathcal{NA}(\mathcal{H})$ and $\mathcal{NA}(\mathcal{H})$-classes refers to the algebra of all norm-attainable operators and the class of all algebras containing norm-attainable operators, respectively.

Definition 2.1 Let $T : \mathcal{NA}(\mathcal{H}) \to \mathcal{NA}(\mathcal{H})$ be defined by

$$T(X) = \sum_{i=1}^{n} A_i X B_i, \ \forall\, X \in \mathcal{NA}(\mathcal{H}),$$

where $A_i,\ B_i$ are fixed in $\mathcal{NA}(\mathcal{H})$. We define the range of T by

$$\mathrm{Ran}T = \{Y \in \mathcal{NA}(\mathcal{H}) : Y = T(X),\ \forall\, X \in \mathcal{NA}(\mathcal{H})\},$$

and the Kernel of T by

$$\mathrm{Ker}T = \{X \in \mathcal{NA}(\mathcal{H}) : T(X) = 0,\ \forall\, X \in \mathcal{NA}(\mathcal{H})\}.$$

It was known [30] that for any of the examples of the elementary operators defined in Section 1 (inner derivation, generalized derivation, basic elementary operator, Jordan elementary