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GLOBAL SOLUTIONS AND FINITE TIME BLOW UP FOR DAMPED KLEIN-GORDON EQUATION*

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Abstract We study the Cauchy problem of strongly damped Klein-Gordon equation. Global existence and asymptotic behavior of solutions with initial data in the potential well are derived. Moreover, not only does finite time blow up with initial data in the unstable set is proved, but also blow up results with arbitrary positive initial energy are obtained.

Key words Klein-Gordon equation; strongly damped; global solutions; blow up

2010 MR Subject Classification 35A01; 35B40; 35B44

1 Introduction

The Klein-Gordon equation is known as one of the nonlinear wave equations arising in relativistic quantum mechanics. On the Cauchy problem of nonlinear Klein-Gordon equation

$$\begin{cases} u_{tt}-\triangle u+u+\gamma u_t-\mu\triangle u_t=|u|^{p-1}u, \quad x\in\mathbb{R}^n,\ t>0,\ \gamma,\mu\geq 0, \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x),\ x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

(H1) $1<p<\infty$ if $n=2$, $1<p<\frac{n+2}{n-2}$ if $n\geq 3$,

(H2) $u_0(x)\in H^1(\mathbb{R}^n)$, $u_1(x)\in L^2(\mathbb{R}^n)$,

many works have been devoted.

For the undamped equation ($\gamma=\mu=0$), the local existence of solutions was proved by Pecher [1]. Then, Levine [2], and Payne and Sattinger [3], and Ball [4] investigated the blow up of solutions for the initial boundary value problem of above equation. In contrast, for sufficiently small initial data, we know that the solutions of the Cauchy problem for the equation global exist in all time (see [5] and the references therein).

For equations with weak damping ($\gamma>0,\mu=0$), Ha and Nakagiri [6] studied the identification problems on an open bounded set Ω with piece wise smooth boundary. Then, Gao and Guo [7] considered the above equations with periodic boundary when $n=2$ and proved the existence and uniqueness of a time-periodic solution thanks to the Galerkin method and

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Leray-Schauder fixed point theorem. While for the numerical solutions, Lin and Cui [8] constructed a model for the above equation and provided a new method based on the reproducing kernel space for solving problem (1.1). As for the global solutions, Nakao and Mitsuhiro [9] derived energy decay estimates to the Cauchy problem (1.1) in $\mathbb{R}^n$. Recently, Ha and Park [10] investigated the existence and uniqueness of solutions by employing the Faedo-Galerkin method and Gronwall's lemma in a noncylindrical domain, then proved the existence and exponential decay rate of the global solutions. Finally, Xu [11] successfully handled the Cauchy problem (1.1) in $\mathbb{R}^n$, moreover, the author employed a new method to prove the blow up of solutions in the case of both $E(0) < d$ and $E(0) = d$, and the asymptotic behavior of global solutions was also derived.

As much less is known for equations with strong damping ($\gamma > 0, \mu > 0$), we refer to the work by Avrin and Joel [12] who considered the above equations in $\mathbb{R}^3$ in the case where the initial data possess radial symmetry, and extended their previous results in [13] which assumed a bounded spatial domain. In particular, the authors constructed a global weak solution v of the undamped equation for higher powers p, which can be approximated arbitrarily closely by the global strong solutions of the above damped equations.

But still many problems remain unsolved, so it is our purpose to shed some further light on problem (1.1). Therefore, in this article, we study the Cauchy problem of nonlinear Klein-Gordon equation in the case of both strong and weak damping ($\gamma > 0, \mu \geq 0$). As problem (1.1) includes the damping terms, the normal convexity method employed in [3] cannot be directly used to derive the finite time blow up of solutions. Therefore, one of the main difficulties of this article is to improve the classical convexity method for proving the finite time blow up of solutions, as well as to obtain a sharp condition of global existence and finite time blow up of solutions for problem (1.1). In addition, as done by Filippo and Marco [14] for damped semilinear wave equations, we will exploit further the properties of the Nehari manifold. In particular, this will enable us to obtain blow up results in correspondence of initial data (u_0, u_1) having arbitrary positive initial energy. As far as we know, this is the first blow up result for problem (1.1) with $E(0) > d$ in the case of weak damping. Finally, we wish to stress that the asymptotic behavior of global solutions indicates the energy of global solutions of problem (1.1) decays exponentially, which is much faster than that in [14].

In what follows, we use the following abbreviations for simplicity of notation:

$$H^1 = H^1(\mathbb{R}^n),\ L^p = L^p(\mathbb{R}^n),\ \|\cdot\|_p = \|\cdot\|_{L^p(\mathbb{R}^n)},\ \|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^n)}.$$

For all $u, v \in H^1$, let $(u, v) = \int uv\mathrm{d}x = \int_{\mathbb{R}^n} uv\mathrm{d}x$ denote the L^2-inner product and put

$$(u, v)_* = \mu \int \nabla u \nabla v \mathrm{d}x + \gamma \int uv\mathrm{d}x, \quad \|u\|_* = (u, u)_*^{\frac{1}{2}}.$$

Throughout the following discussion, we may consider the C^1 Nehari functionals $J(u), I(u) : H^1(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \qquad I(u) = \|u\|_{H^1}^2 - \|u\|_{p+1}^{p+1},$$

and introduce the stable set W, unstable set V, and the so-called Nehari manifold $\mathcal{N}$, respectively, defined by

$$W = \{u \in H^1 | I(u) > 0, J(u) < d\} \cup \{0\},$$