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A NOTE ON GRADIENT BLOWUP RATE OF THE INHOMOGENEOUS HAMILTON-JACOBI EQUATIONS*

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Abstract The gradient blowup of the equation $u_t = \Delta u + a(x)|\nabla u|^p + h(x)$, where $p > 2$, is studied. It is shown that the gradient blowup rate will never match that of the self-similar variables. The exact blowup rate for radial solutions is established under the assumptions on the initial data so that the solution is monotonically increasing in time.

Key words Gradient blowup; Hamilton-Jacobi equation; inhomogeneous

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1 Introduction

In this article, we mainly study the following parabolic equation with the nonlinear gradient source

$$\begin{cases} u_t = \Delta u + a(x)|\nabla u|^p + h(x), & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where $p > 2$, $a(x) \in C^2(\overline{\Omega})$, $0 < a \le a(x) \le b < \infty$, and $h(x) \in C^1(\overline{\Omega})$, $\Omega \subset \mathbb{R}^N (N \ge 2)$ being a bounded smooth domain. Moreover, the nonnegative function u_0 satisfies

$$u_0(x) \in BC^2(\overline{\Omega}) \quad \text{and} \quad u_0(x) = 0 \ \text{ for } \ x \in \partial\Omega. \tag{1.2}$$

The space BC^2 is the set of all functions with bounded and continuous derivatives up to order 2. In the radial symmetric case with $\Omega = B_1 = \{x| \ |x| < 1\}$, $a(x) = a(r)$, $h(x) = h(r)$ and $u_0(x) = u_0(r)$ with $r = |x|$, by a direct computation, then the radial solution $u(r,t)$ satisfies

$$\begin{cases} u_t = u_{rr} + \dfrac{N-1}{r}u_r + a(r)|u_r|^p + h(r), & r \in (0,1), t > 0, \\ u_r(0,t) = 0, \quad u(1,t) = 0, & t > 0, \\ u(r,0) = u_0(r), & r \in [0,1], \end{cases} \tag{1.3}$$

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where $p > 2$, $a(r) \in C^2[0,1]$, $0 < a \le a(r) \le b < \infty$, $h(r) \in C^1[0,1]$, and the nonnegative function $u_0(r)$ satisfies the compatibility condition

$$u_0(r) \in BC^2[0,1], \quad u_0(1) = 0. \tag{1.4}$$

The equation

$$u_t = \Delta u + a(x)|\nabla u|^p + h(x)$$

is a simple example of the parabolic equation with a nonlinearity depending on the first order spatial derivatives of u. Generally, the equation $u_t - \epsilon\Delta u = f(x, \nabla u)$ may be viewed as the viscosity approximation (as $\epsilon \to 0^+$) of Hamilton-Jacobi type equations from stochastic control theory (see [6]). Also the equation appears in the physical theory of growth and roughening of surfaces, where it is known as the Kardar-Parisi-Zhang equation [3, 4].

Finite time blowup phenomena have attracted a lot of attention in the past few years. Many studies concentrated on the blowup of solution itself. These include blowup criteria, blowup locations, blowup rates, and blowup profiles. In particular, for the classical semilinear heat equation $u_t = \Delta u + u^p$, and the degenerate parabolic equations such as $u_t = \Delta u^m + u^p$, $u_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u) + u^q$, we refer the readers to [7, 13] and the references therein. For quasilinear equation of the type

$$u_t = \Delta u + F(x, t, u, \nabla u)$$

with Dirichlet boundary condition, it is well known (cf. [5, 7]) that the derivatives of any bounded solution are also uniformly bounded if

$$|F(x, t, u, \xi)| \le C(u)(1 + |\xi|^2).$$

Moreover, it is well known that under certain conditions $|\nabla u|$ blows up in a finite time $t = T$, that is,

$$\limsup_{t\to T^-} \{\sup_{x\in\overline{\Omega}} |\nabla u(x,t)|\} = \infty,$$

while the solutions themselves are uniformly bounded. This blowup phenomenon is called gradient blowup. For the gradient blowup, most previous works were about the blowup criteria, that is, when the gradient blowup occurs. Little is known about the blowup rates. See, for example, [2, 9, 10]. The main purpose of this article is to investigate the blowup rate of the model problem (1.1).

For the problem $u_t = u_{xx} + u_x^p$, Conner and Grant [1] have proved that

$$\sup_{\overline{\Omega}\times[0,t]} |\nabla_x u(x,\tau)| \ge c(T-t)^{-1/(p-2)}$$

under the assumption $u_0' > 0$ in Ω (where $\Omega = (0,1)$). In addition, on the basis of some numerical simulations, they deduced that $(T-t)^{-1/(p-2)}$ should be the exact gradient blowup rate for the one dimensional problem. Guo and Hu [2] have provided a rigorous proof of this result under the assumption that the solution is monotonically increasing in t. Their proof is different and much simple, and they don't need the additional assumption $u_0' > 0$ in $(0,1)$. Recently, in the article [9], Zhang and Hu have generalized the result to the equation $u_t = u_{xx} + x^m u_x^p$ by a very different technique from [2]. In another article [10], Zhang and Hu studied the problem $u_t = u_{xx} + e^{u_x}$ and also obtained the non-self-similar gradient blowup rate