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INTRINSIC EQUATIONS FOR A GENERALIZED RELAXED ELASTIC LINE ON AN ORIENTED SURFACE IN THE GALILEAN SPACE*

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Abstract In this article, we derive the intrinsic equations for a generalized relaxed elastic line on an oriented surface in the Galilean 3-dimensional space G_3. These equations will give direct and more geometric approach to questions concerning about generalized relaxed elastic lines on an oriented surface in G_3.

Key words Galilean space; generalized relaxed elastic line; variational problem; intrinsic formulation; geodesic

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1 Introduction

Elasticity theory, being a direct descendant of differential calculus, is certain to have a great deal of interest to say about geometry, analysis, physics, chemistry, and engineering. Therefore, several geometers were interested in studying the elasticity theory in Euclidean Space [1–7]. Nevertheless, similar applications of this theory in Minkowski space can be found in [8–10]. The main point of the studying the elasticity theory is defining the intrinsic equations for a relaxed elastic line on an oriented surface.

The natural variational integrals in geometry are the common integrals on space curves $\alpha(s)$. These include the length $L(\alpha) = \int \mathrm{d}s$, total squared curvature $K(\alpha) = \int \kappa^2 \mathrm{d}s$ used in [4, 5], total squared torsion $T(\alpha) = \int \tau^2 \mathrm{d}s$ used in [6, 10] and the integral $H(\alpha) = \int \kappa^2 \tau \mathrm{d}s$ used in [2]. An elastic line of length l is defined as a curve with associated energy equation $K = \int_0^l \kappa^2(s)\,\mathrm{d}s$ by Nickerson and Manning in [5], where s is the arc-length along the curve, $\kappa^2(s)$ being the square curvature in there. The integral K is called the total square curvature. If no boundary conditions are imposed at $s = l$, and if no external forces act at any s, the elastic line is relaxed [4]. The trajectory of a relaxed elastic line in a space and on a plane is a straight line because the positive indefinite quantity that defines K takes its minimum value of zero when the square curvature vanishes for all s. The trajectory of a relaxed elastic line constrained to lie on a general surface is, which in general bounds, the possible values of K away from zero.

Hilbert and Cohn-Vossen stated in [11] that a relaxed elastic line with specified position and tangent at $s = 0$ always has the trajectory of a geodesic. However, Nickerson and Manning

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have proved in [4, 5] that the conslusion of Hilbert and Cohn-Vossen is incorrect. Nickerson and Manning in [5] have obtained intrinsic equations of the relaxed elastic line on an oriented surface as a model of DNA molecule. As briefly mentioned above, the authors calculated the intrinsic equations of relaxed elastic lines with the aid of K_g just by using curvature of an elastic curve in [5]. In [1, 2, 6–10], similar equations for similar energy functions in Euclidean space, Pseudo-Euclidean space, and Minkowski space are shown.

Besides Euclidean Geometry, a range of new types of geometries have been invented and developed in the last two centuries. They can be introduced in a variety of ways. One possible way is through projective manner, where one can express metric properties through projective relations. Among these geometries, there is also Galilean geometry, which is our concern here.

As the Darboux frames are different by the signs in articles mentioned above, the intrinsic equations of relaxed elastic lines are different from each other by their signs. However, in Galilean space, we have different intrinsic equations of relaxed elastic lines due to the structure of Darboux frame. For example, in Lemma 3.1, one can see that $\frac{\mathrm{d}\lambda}{\mathrm{d}t}|_{t=0} = 0$ for Galilean space whereas $\frac{\mathrm{d}\lambda}{\mathrm{d}t}|_{t=0} = \pm\int_0^l \mu\kappa_g \mathrm{d}s$ in Euclidean and Minkowski spaces. In the literature, there is not any study on the elasticity problem in the Galilean space G_3 which is our main focus in this article. To this purpose, we first derive the intrinsic equations for a generalized relaxed elastic line on an oriented surface in G_3 by using the notion of Galilean Darboux frame of a curve in G_3. Then, we obtain intrinsic equations for a generalized relaxed elastic line on an oriented surface in G_3 using this frame. Here, as the energy density is given as a function $f(\kappa,\tau) = \kappa^2\tau$, we give the equations in G_3 with the aid of K_n using both the curvature and the torsion of the elastic curve.

The main result of this article is given in Theorem 3.2. We describe the geometric meaning of this theorem in Section 4.

2 Preliminaries on Galilean Geometry

"All geometry is projective geometry" (A. Cayley). From Cayley's point of view, G_3 is a real 3-dimensional projective space $P^3(\mathbb{R})$, the set of equivalent classes of $\sim$ on $\mathbb{R}^4 - \{0\}$ by equivalence relation $x \sim y$ iff $x = \lambda y$ for some $\lambda \in \mathbb{R} - \{0\}$. Thus, $P^3(\mathbb{R})$ is obtained as a factor space on $\mathbb{R}^4 - \{0\}$ by $\sim$, that is, $P^3(\mathbb{R}) \cong (\mathbb{R}^4 - \{0\})/\sim$ [12]. We can think of $P^3(\mathbb{R})$ more geometrically as a set of lines through the origin in $\mathbb{R}^4$. G_3 is a real Cayley-Klein space equipped with the projective metric of signature $(0,0,+,+)$, as shown in [13]. The absolute of the Galilean geometry is an ordered triple $\{w,f,I\}$, where w is the ideal (absolute) plane, f is the line (absolute line) in w, and I is the fixed elliptic involution of points of f. The points, the lines, and the planes of $P^3(\mathbb{R})$ are the one-dimensional, two-dimensional, and three-dimensional subspaces of $\mathbb{R}^4$, respectively [14]. Therefore, G_3 contains $\mathbb{R}^3$ as a proper subset, and the complement in G_3 to w is diffeomorphic to $\mathbb{R}^3$.

Let P be any point of $\mathbb{R}^3$ with affine coordinates (x,y,z). Write (x,y,z) as $(\frac{X_1}{X_0}, \frac{X_2}{X_0}, \frac{X_3}{X_0})$, where X_0 is some common deminator. Call (X_0, X_1, X_2, X_3) the homogeneous coordinates of P. Thus, the homogeneous coordinates $(X_0 : X_1 : X_2 : X_3)$ and $\rho(X_0 : X_1 : X_2 : X_3)$ refer to the same point for all $\rho \in \mathbb{R} - \{0\}$, [14]. Now, we can introduce homogeneous coordinates in G_3 in such a way that the absolute plane w is given by $X_0 = 0$, the absolute line f by $X_0 = X_1 = 0$, and the elliptic involution I by $(0:0:X_2:X_3) \to (0:0:X_3:-X_2)$. In affine coordinates, the