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EXISTENCE OF SOLUTIONS OF NONLINEAR FRACTIONAL PANTOGRAPH EQUATIONS*

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Abstract This article deals with the existence of solutions of nonlinear fractional pantograph equations. Such model can be considered suitable to be applied when the corresponding process occurs through strongly anomalous media. The results are obtained using fractional calculus and fixed point theorems. An example is provided to illustrate the main result obtained in this article.

Key words Fractional differential equations; pantograph equations; fixed point theorems

2010 MR Subject Classification 34A08

1 Introduction

Recently fractional differential equations emerged as a new branch of applied mathematics, which has been used for many mathematical models in science and engineering [2, 13, 24, 33]. In fact, fractional differential equations is considered as an alternative model to nonlinear differential equations [9]. Theory of fractional differential equations has been extensively studied by many authors [1, 19, 20, 26]. Balachandran et al [3–5, 7, 8] studied the existence of solutions of different types of fractional differential equations whereas Hernandez et al [27] discussed the recent developments in the theory of fractional differential equations. Many models are reformulated and expressed in terms of fractional differential equations so that the physical meaning will be incorporated in the mathematical models more realistically.

It is well known that in the deterministic situation there is a very special delay differential equation known as the pantograph equation

$$y'(t) = ay(t) + by(\lambda t), \quad 0 \le t \le T,$$

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$$y(0) = y_0,$$

where $0 < \lambda < 1$. It arises in quite different fields of pure and applied mathematics such as number theory, dynamical systems, probability, quantum mechanics, and electro dynamics. In particular, it is used by Ockendon and Taylor [28] to study how the electric current is collected by the pantograph of an electric locomotive, from where it gets its name. Properties of the analytic solution of this equation as well as numerical methods have been studied by several authors [18, 21, 22]. Iserels and his coworkers [12, 14–16] studied extensively on the ordinary pantograph equations. Recently multi-pantograph equation of the form

$$u'(t) = au(t) + \sum_{i=1}^{m} \mu_i(t)u(\lambda_i t) + f(t), \quad t \geq 0$$

received considerable interest among the researchers [22, 31, 34]. The generalized nonlinear multi-pantograph equation of the form

$$\begin{aligned} u'(t) &= f(t, u(t), u(\lambda_1 t), \cdots, u(\lambda_m t)), \quad 0 \leq t \leq T, \\ u(0) &= u_0, \end{aligned}$$

has been discussed by Liu and Li [23], where as the nonlinear neutral pantograph equation

$$\begin{aligned} u'(t) &= F(t, u(t), u(\lambda t), u'(\lambda t)), \quad t > 0, \\ u(0) &= u_0, \end{aligned}$$

was investigated by Sezer et al [31]. Due to its importance in many applied fields, it is interesting to study the fractional model of the pantograph equations. Such model can be considered suitable to be applied when the corresponding process occurs through strongly anomalous media. In this article, we establish the existence of solutions of abstract fractional pantograph equations using the fractional calculus and fixed point theorems.

2 Preliminaries

We need some basic definitions and properties of fractional calculus that are used in this article. Let X be a Banach space and $C(J, X)$ be the Banach space of continuous functions $x(t)$ with $x(t) \in X$ for $t \in J := [0, T]$ and $\|x\| = \max_{t \in J} |x(t)|$. In addition, $B_r(x, X)$ represents the closed ball with center at x and radius r in X.

Definition 2.1 The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f \in L_1(R_+)$, is defined as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n - 1 < \alpha < n, n \in N$, is defined as

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) \mathrm{d}s,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.