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NEW EXISTENCE RESULTS OF POSITIVE SOLUTION FOR A CLASS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS*

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Abstract In this article, the existence and uniqueness of positive solution for a class of nonlinear fractional differential equations is proved by constructing the upper and lower control functions of the nonlinear term without any monotone requirement. Our main method to the problem is the method of upper and lower solutions and Schauder fixed point theorem. Finally, we give an example to illuminate our results.

Key words Fractional differential equations; positive solution; upper and lower solutions; existence and uniqueness; fixed point theorem

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1 Introduction

We consider the following nonlinear fractional differential equation

$$D^{\alpha}u = f(t,u),\ 0 < t < 1, u(0) = 0, \tag{1}$$

where $0 < \alpha < 1$, D^{α} is the Riemann-Liouville fractional derivative defined as follow

$$D^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\alpha}u(s)\mathrm{d}s, \tag{2}$$

where Γ denotes the Gamma function, and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is a given continuous function.

Fractional differential equation (1) can be extensively applied to the various physics, mechanics, chemistry, and engineering etc., see [1–3]. Hence, in recent years, fractional differential equations have been of great interest and there have been many results on existence and uniqueness of the solutions of FDE. D.Delbos and L.Rodino proved the existence of the solutions to

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the nonlinear fractional equation (1) using Banach contraction principle and Schauder fixed point theorem respectively [4]; Shuqin Zhang obtained the existence and uniqueness of positive solution utilizing the method of the upper and lower solution and cone fixed-point theorem [5]; Qingliu Yao considered the existence of positive solution to the fractional equation controlled by the power function employing Krasnosel's kii fixed-point theorem of cone expansion-compression type [6]. Recently, V. Lakshmikantham obtained the existence of the local and global solutions to (1) using classical differential equation theorem [7]. However, in the previous works, the nonlinear term has to satisfy the monotone or others control conditions. In fact, the fractional differential equations with nonmonotone function can respond better to impersonal law, so it is very important to weaken monotone condition. In this article, we mainly investigated the fractional differential equations without any monotone requirement on nonlinear term by constructing upper and lower control functions and exploiting upper and lower solutions method and Schauder fixed-point theorem. The existence and uniqueness of positive solution for equation (1) is obtained. This work is motivated from the references [5, 8]. Other related results on the fractional differential equations can be found in references [9–25].

This article is organized as follows. In Section 2, we consider the existence of positive solution for Eq. (1) utilizing the upper and lower solutions method and Schauder fixed-point theorem. Section 3 contains result for uniqueness of a positive solution.

2 Existence of Positive Solution

Let $X = C[0,1]$ be the Banach space endowed with the maximum norm and define

$$K = \{u \in X : u(t) \geq 0,\ 0 \leq t \leq 1\}.$$

The positive solution which we consider in this article is such that $u(0) = 0$, $u(t) > 0$, $0 < t \leq 1$, $u(t) \in X$. According to [4, Proposition 2.4], Eq. (1) is equivalent to the integral equation

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) \mathrm{d}s,\ 0 \leq t \leq 1, \tag{3}$$

where Γ denotes the Gamma function. The integral equation (3) is also equivalent to fixed point equation $Tu(t) = u(t)$, $u(t) \in C[0,1]$, where operator $T : K \to K$ is defined as

$$(Tu)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) \mathrm{d}s,\ \ 0 \leq t \leq 1, \tag{4}$$

then we have the following lemma.

Lemma 2.1 The operator $T : K \to K$ is compact.

Proof The operator $T : K \to K$ is continuous in view of the assumption of nonnegativeness and continuity of $f(t,u)$.

Let $M \subset K$ be bounded, that is, there exists a positive constant l such that $\|u\| \leq l$ for any $u \in M$, and let $L = \max\limits_{0 \leq t \leq 1, 0 \leq u \leq l} f(t, u(t)) + 1$, then, for any $u \in M$, we have

$$\begin{aligned} |Tu(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) \mathrm{d}s \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f(s, u(s)) \right| \mathrm{d}s \end{aligned}$$