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ON GROUND STATE SOLUTIONS FOR SUPERLINEAR DIRAC EQUATION*

Jian ZHANG(张健) *Xianhua TANG* (唐先华) *Wen ZHANG* (张文)

School of Mathematics and Statistics, Central South University, Changsha 410083, China

E-mail: *zhangjian433130@163.com; tangxh@mail.csu.edu.cn; zwmath2011@163.com*

Abstract This article is concerned with the nonlinear Dirac equations

$$-\mathrm{i}\partial_t\psi = \mathrm{i}c\hbar\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc^2\beta\psi + R_\psi(x,\psi) \ \text{ in } \ \mathbb{R}^3.$$

Under suitable assumptions on the nonlinearity, we establish the existence of ground state solutions by the generalized Nehari manifold method developed recently by Szulkin and Weth.

Key words Nonlinear Dirac equation; ground state solutions; generalized Nehari manifold; strongly indefinite functionals

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1 Introduction

We study the following nonlinear Dirac equation

$$-\mathrm{i}\partial_t\psi = \mathrm{i}c\hbar\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc^2\beta\psi + R_\psi(x,\psi), \tag{1.1}$$

where $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, $\partial_k=\frac{\partial}{\partial x_k}$, $R:\mathbb{R}^3\times\mathbb{C}^4\to\mathbb{R}$ satisfies $R(x,e^{\mathrm{i}\theta}\psi)=R(x,\psi)$ for all $\theta\in\mathbb{R}$, $\psi\in\mathbb{C}^4$ represents the wave function of the state of an electron, c denotes the speed of light, $m>0$, the mass of the electron, $\hbar$ is Planck's constant, and $\alpha_1,\alpha_2,\alpha_3$, and β are the 4×4 complex matrices:

$$\beta=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\quad \alpha_k=\begin{pmatrix} 0 & \sigma_k\\ \sigma_k & 0\end{pmatrix},\quad k=1,2,3,$$

with

$$\sigma_1=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}.$$

Such problem arises in the study of stationary solutions to the nonlinear Dirac equation which models extended relativistic particles in external fields and was used as effective theories in atomic, nuclear, and gravitational physics [1]. We are going to look for stationary solutions of

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(1.1), that is, solutions of the form $\psi(t,x)=e^{\frac{i\theta t}{\hbar}}u(x)$. Then, $\psi(t,x)$ is a solution of (1.1) if and only if u satisfies

$$-\mathrm{i}c\hbar\sum_{k=1}^{3}\alpha_k\partial_k u+mc^2\beta u=R_u(x,u)-\theta u. \tag{1.2}$$

For simplicity, we rewrite (1.2) as

$$-\mathrm{i}\sum_{k=1}^{3}\alpha_k\partial_k u+a\beta u+\omega u=G_u(x,u), \tag{1.3}$$

where $a>0$ and $\omega\in\mathbb{R}$. The problem (1.3) was extensively investigated in the literatures based on various assumptions on the nonlinearity $G(x,u)$, and the existence and multiplicity of stationary solutions of several models of particle physics were established. In [11–13] and [14], the so-called Soler potential was considered by using shooting methods. To our knowledge, the variational method was first used in [15] for Soler model. For a survey, see [16].

In this article, we will consider the nonlinearity of the type

$$G(x,u)=-\frac{1}{2}V(x)u\tilde{u}+F(x,u).$$

Then, the problem (1.3) reads as

$$-\mathrm{i}\sum_{k=1}^{3}\alpha_k\partial_k u+(V(x)+a)\beta u+\omega u=F_u(x,u). \tag{1.4}$$

In this article [3], Bartsch and Ding studied the periodic Dirac equation (1.4) by critical point theory for strongly indefinite problem recently established in [2]. By assuming $V(x)$ and $F(x,u)$ were periodic in x and some conditions were weaker than the Ambrosetti-Rabinowitz condition, the authors first established the analytic setting for the problem and then obtained the existence of stationary solutions. When $F(x,u)$ is even in u, then (1.4) possesses infinitely many geometrically different solutions. Later, without the assumption that $F(x,u)$ is even in u, Zhao and Ding [19] also obtained infinitely many geometrically different solutions for the periodic Dirac equation (1.4) with superquadratic and asymptotically quadratic nonlinearities by using a reduction method. This idea can be tracked to the work in [31]. In [22], under the periodic assumption, Yang and Ding obtained the existence of ground state solutions by using variant generalized weak linking theorem for a strongly indefinite problem developed by Schechter and Zou [32].

If the potential is non-periodic, a lot of articles appeared dealing with the existence and multiplicity of stationary solutions. An asymptotically quadratic nonperiodic problem with Coulomb-type potential was considered in [4], and in [6] Ding and Wei treated the superquadratic subcritical nonlinearities with mainly the limits of the potential and the nonlinearity existing as $|x|\to\infty$, and the Ambrosetti-Rabinowitz condition also plays an important role. Under the stronger condition on the potential, the authors in [17] also considered the nonperiodic case with asymptotically quadratic and superquadratic nonlinearities. By using the critical point theory in [2], they established the existence and multiplicity of solutions. In [18], Zhang et al proved the existence of ground state solutions by the generalized Nehari manifold. For singularly perturbation problem and concentration phenomena of semi-classical solution, we refer readers to [5, 7–9] and references therein.