Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2014,**34B**(3):891–904
Mathematica Acta Scientia
数学物理学报
http://actams.wipm.ac.cn

BOUNDEDNESS OF STEIN'S SQUARE FUNCTIONS ASSOCIATED TO OPERATORS ON HARDY SPACES*

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Abstract Let (X, d, μ) be a metric measure space endowed with a metric d and a nonnegative Borel doubling measure μ. Let L be a second order non-negative self-adjoint operator on $L^2(X)$. Assume that the semigroup e^{-tL} generated by L satisfies the Davies-Gaffney estimates. Also, assume that L satisfies Plancherel type estimate. Under these conditions, we show that Stein's square function $\mathcal{G}_\delta(L)$ arising from Bochner-Riesz means associated to L is bounded from the Hardy spaces $H^p_L(X)$ to $L^p(X)$ for all $0 < p \leq 1$.

Key words Stein's square function; non-negative self-adjoint operator; Hardy spaces; Davies-Gaffney estimate; Plancherel type estimate

2010 MR Subject Classification 42B15; 42B25; 42F30

1 Introduction and Main Results

Let (X, d, μ) be a metric measure space endowed with a metric d and a nonnegative Borel doubling measure μ satisfying the doubling condition, that is, there exists a constant $C > 0$ such that for any $x \in X$ and for any $r > 0$,

$$V(x, 2r) \leq CV(x, r) < \infty, \tag{1.1}$$

where $B(x, r) = \{y \in X : d(x, y) < r\}$ and $V(x, r) = \mu(B(x, r))$. In particular, X is a space of homogeneous type. A more general definition and further studies of these spaces can be found in [11, Chapter 3].

Note that the doubling property implies the following strong homogeneity property,

$$V(x, \lambda r) \leq C\lambda^n V(x, r) \tag{1.2}$$

for some $C, n > 0$ uniformly for all $\lambda \geq 1$ and $x \in X$. The smallest value of the parameter n is a measure of the dimension of the space. There also exist C and D so that

$$V(y, r) \leq C\Big(1 + \frac{\mathrm{d}(x, y)}{r}\Big)^D V(x, r) \tag{1.3}$$

uniformly for any $x, y \in X$ and $r > 0$. Indeed, property (1.3) with $D = n$ is a direct consequence of the triangle inequality for the metric d and the strong homogeneity property (1.2). When X is Ahlfors regular, that is, $V(x, r) \sim r^n$ uniformly in x, the value D can be taken to be 0.

*Received June 28, 2013; revised September 29, 2013.

The following shall be assumed throughout this article unless otherwise specified:

(H1) L is a non-negative self-adjoint operator on $L^2(X)$;

(H2) The operator L generates an analytic semigroup $\{e^{-tL}\}_{t>0}$ on $L^2(X)$ which satisfies the Davies-Gaffney estimate. That is, there exist constants C, $c>0$ such that for any open subsets U_1, $U_2\subset X$,

$$|\langle e^{-tL}f_1,f_2\rangle|\le C\exp\Big(-\frac{\mathrm{dist}(U_1,U_2)^2}{c\,t}\Big)\|f_1\|_{L^2(X)}\|f_2\|_{L^2(X)},\quad \forall\, t>0, \tag{1.4}$$

for every $f_i\in L^2(X)$ with $\operatorname{supp} f_i\subset U_i$, $i=1,2$, where $\mathrm{dist}(U_1,U_2):=\inf_{x\in U_1,y\in U_2} d(x,y)$.

Examples of families of operators for which condition (1.4) holds include semigroups generated by second order elliptic self-adjoint operators in divergence form on the Euclidean spaces $\mathbb{R}^n$, Schrödinger operators with real potentials and magnetic field (see, for example, [8, 13, 14, 16–19, 23, 24, 26, 27, 29]).

As L is a non-negative self-adjoint operator acting on $L^2(X)$, it admits a spectral resolution

$$L=\int_0^\infty \lambda \mathrm{d}E(\lambda).$$

For a complex number $\delta=\sigma+\mathrm{i}\tau,\sigma>-1$, we can define the Bochner-Riesz mean $S_R^\delta(L)=\big(I-L/R^2\big)_+^\delta$ of order δ of a function f as

$$S_R^\delta(L)f(x)=\int_0^R\left(1-\frac{\lambda}{R^2}\right)^\delta \mathrm{d}E(\lambda)f(x),\quad x\in X \tag{1.5}$$

using the spectral theorem. We then consider the following square function associated to an operator L

$$\mathcal{G}_\delta(L)f(x)=c_\delta\left(\int_0^\infty\Big|\frac{\partial}{\partial R}S_R^{\delta+1}(L)f(x)\Big|^2R\mathrm{d}R\right)^{1/2},\quad x\in X, \tag{1.6}$$

where $c_\delta=\frac{1}{2(\delta+1)}$.

Note that when L is the usual Laplacian $-\Delta$ on $\mathbb{R}^n$, the square function $\mathcal{G}_\delta(\Delta)$ is introduced by E.M. Stein in his study of Bochner-Riesz means [30]. It is known that the L^p boundedness of $\mathcal{G}_\sigma(\Delta)$ for $1<p\le 2$ holds if and only if $\sigma>n(1/p-1/2)-1/2$ (see [20, 21, 30]). For the range $p>2$, the condition $\sigma>\max\{1/2,n(1/2-1/p)\}-1$ is known to be necessary and sufficient in the dimensions $n=1$ and 2. In dimensions $n\ge 3$, there are some partial results, see, for instances, for $\sigma>n(1/2-1/p)-1/2$ in [20, 21]. For $0<p\le 1$, if $\sigma>n\big(\frac1p-\frac12\big)-\frac12$, then $\mathcal{G}_\sigma(\Delta)$ is bounded from H^p to L^p [25]. Boundedness of the square function $\mathcal{G}_\delta(\Delta)$ was studied extensively because of its important role in the Bochner-Riesz analysis and we refer the reader to [9, 20, 21, 25, 30] and the references therein.

Recently, P. Chen, X.T. Duong, and L.X. Yan studied the L^p boundedness of Stein's square function $\mathcal{G}_\delta(L)$ when the semigroup e^{-tL}, generated by $-L$ on $L^2(X)$, has the kernel $p_t(x,y)$ that satisfies the following Gaussian upper bound

$$\big|p_t(x,y)\big|\le\frac{C}{V(x,t^{1/2})}\exp\Big(-\frac{d(x,y)^2}{c\,t}\Big). \tag{1.7}$$

They showed that under the assumption of Plancherel type estimate of L, that is, for some $2\le q\le\infty$ and any $t>0$ and all Borel functions F such that $\operatorname{supp} F\subseteq[0,t]$,

$$\int_X|K_{F(\sqrt{L})}(x,y)|^2\mathrm{d}\mu(x)\le\frac{C}{V(y,t^{-1})}\|F(t\cdot)\|_{L^q}^2, \tag{1.8}$$