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GLOBAL EXISTENCE, UNIFORM DECAY AND EXPONENTIAL GROWTH FOR A CLASS OF SEMI-LINEAR WAVE EQUATION WITH STRONG DAMPING*

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Abstract In this paper, we consider the nonlinearly damped semi-linear wave equation associated with initial and Dirichlet boundary conditions. We prove the existence of a local weak solution and introduce a family of potential wells and discuss the invariants and vacuum isolating behavior of solutions. Furthermore, we prove the global existence of solutions in both cases which are polynomial and exponential decay in the energy space respectively, and the asymptotic behavior of solutions for the cases of potential well family with $0 < E(0) < d$. At last we show that the energy will grow up as an exponential function as time goes to infinity, provided the initial data is large enough or $E(0) < 0$.

Key words strong damping; nonlinear damping; global existence; polynomial decay; exponential decay; exponential growth

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following problem

$$\begin{cases} u_{tt} - \triangle u - \omega \triangle u_t + a|u_t|^{m-2}u_t = b|u|^{p-2}u, & x \in \Omega, \ \ t > 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), & x \in \Omega, \\ u(t,x) = 0, & x \in \partial\Omega, \ \ t > 0, \end{cases} \tag{1.1}$$

where $\omega \geq 0, a > 0, b > 0, m \geq 2, p > 2$ and $\triangle$ is Laplacian in $\mathbb{R}^N$.

In the case of $m = 2$, problem (1.1) was studied by Gazzola and Squassina [4], they proved the global existence and polynomial decay property of solutions of problem (1.1) provided that

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the initial data are in the potential well. The proof of decay property is based on the method used in [8] and [9], where the authors obtained the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}[(1+t)E(u(t))] \leq E(u(t)), \tag{1.2}$$

where $E(u(t))$ is the energy of the solutions. Recently, for $m = 2$, Gerbi and Houari [5] obtained the exponential decay based on a small perturbation of energy. In the absence of the nonlinear source term, it was well known that the presence of one damping term (i.e., $\omega > 0$ or $a > 0$) ensure global existence and decay of solution for arbitrary initial data (see [7, 11]). For $\omega = a = 0$, the nonlinear term $|u|^{p-2}u$ cause finite-time blow-up of solutions with negative initial energy (see [2]).

In the absence of the strong damping term (i.e., $\omega = 0$), the interaction between the weak damping term and the source term were considered by many authors. Levine [12, 13] showed that solutions with negative initial energy blow up in finite time when $m = 2$, i.e., the weak damping term is linear. When the linear term u_t is replaced by $|u_t|^{m-2}u_t$, Georgiev and Todorova [6] extended Levine's result to the case $m > 2$, and introduced a method different from the one known as the concavity method used by Levine. They determined suitable relations between m and p, for which there is global or alternatively finite time blow-up, precisely, they showed the solutions globally in time if $m \geq p$ and blow-up in finite time if $m < p$ and the initial energy is sufficiently negative. Vitillaro [23] extended the result in the situations where the solution has positive energy. Similar results were established by Todorova [22] for the Cauchy problem.

We recall here that the potential well introduced by Payne and Sattinger [19, 20] is also useful and widely used to investigate the problem (1.1) with $\omega = 0$, see [3, 8, 9, 22, 23].

Introducing a strong damping term $\triangle u_t$ make the problem differently from the one considered in [6]. For this reason, less results are at present time known for the wave equation with strong damping term and many problem remain unsolved, see Gazzola and Squassina [4] even for the case $m = 2$.

The purpose of this paper is to investigate e problem (1.1) for the case $\omega \geq 0$ and $m \geq 2$, which is an extension of [4]. Firstly by the fixed point theory, we establish the local existence and uniqueness for problem (1.1); then we use a family of potential wells introduced by Liu [15, 16], which include the well known potential well as a special case. By using this new method, the authors not only obtained some new result on global existence and invariant sets of solutions, but also discovered the vacuum isolating of solutions. Recently this method was extended by Liu [17] and Xu [24] to study the Cauchy problem of nonlinear Klein-Gordon equations with $\omega = 0$ and $m = 2$. In this paper we are also interested in the similar properties of the solutions of the problem. Motivated by [1] we intend to study the polynomial asymptotic behavior and by [3, 5] we intend to study the exponential asymptotic as well. In the presence of the strong damping term (i.e., $\omega \neq 0$), there are few results about the blow-up in finite time. Recently, Song and Zhong [21] showed the blow-up in finite time by the known concavity method used by Levine, only on $\omega = b = 1$, and $a = 0$. In this paper we will also consider the blow-up property in infinity time, i.e., exponential growth.

Our paper is organized as follows. In Section 2 we present the local existence and uniqueness for solution of problem (1.1). In Section 3, we introduce a family of potential wells and