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EXISTENCE OF MULTIPLE SOLUTIONS FOR SINGULAR QUASILINEAR ELLIPTIC SYSTEM WITH CRITICAL SOBOLEV-HARDY EXPONENTS AND CONCAVE-CONVEX TERMS*

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Abstract The main purpose of this paper is to establish the existence of multiple solutions for singular elliptic system involving the critical Sobolev-Hardy exponents and concave-convex nonlinearities. It is shown, by means of variational methods, that under certain conditions, the system has at least two positive solutions.

Key words singular elliptic system; concave-convex nonlinearities; positive solution; Nehari manifold; critical Sobolev-Hardy exponent

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1 Introduction

Let $0 \in \Omega \subset R^N$ be a bounded domain with the smooth boundary $\partial\Omega$. We investigate the multiplicity of positive solutions to the following quasilinear elliptic problem:

$$\begin{cases} -\Delta_p u - l\dfrac{|u|^{p-2}u}{|x|^p} = \lambda f(x)\dfrac{|u|^{q-2}u}{|x|^s} + \dfrac{2\alpha}{\alpha+\beta}h(x)\dfrac{|u|^{\alpha-2}u|v|^\beta}{|x|^t} & \text{in } \Omega, \\ -\Delta_p v - l\dfrac{|v|^{p-2}v}{|x|^p} = \mu g(x)\dfrac{|v|^{q-2}v}{|x|^s} + \dfrac{2\beta}{\alpha+\beta}h(x)\dfrac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^t} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $1 < p < N$, $1 < q < p$, $\lambda,\ \mu > 0$, $0 \le l < \bar{l}$, $0 \le s,\ t < p$, $\alpha > 1$, $\beta > 1$ satisfying $\alpha + \beta = p^*(t)$, f, g and h are nonnegative

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functions, $p^*(t) = \frac{p(N-t)}{N-p}$ is the so-called critical Sobolev-Hardy exponent and $\bar{l} = (\frac{N-p}{p})^p$ is the best Hardy constant. Note that $p^*(0) = p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent.

Set $f(x) = g(x) = h(x) = 1,\ \lambda = \mu,\ u = v,\ \alpha = \beta = \frac{r}{2}$, then system (1.1) reduces to the following scalar quasilinear elliptic problem:

$$\begin{cases} -\Delta_p u - l\dfrac{|u|^{p-2}u}{|x|^p} = \lambda\dfrac{|u|^{q-2}u}{|x|^s} + \dfrac{|u|^{r-2}u}{|x|^t} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $r = p^*(t) = \frac{p(N-t)}{N-p}$. For the case $p = 2$, much effort was devoted to the study of the existence of solutions for singular semilinear elliptic problems with critical Sobolev exponents or critical Sobolev-Hardy exponents, see for example [1–5] and [6–8]. With regard to $p \neq 2$, many interesting results were also obtained. Kang [9] considered problem (1.2) and proved the existence of solutions with sublinear perturbation of $p \leq q < p^*(s)$. Subsequently, The author of [10] studied the existence of multiple positive solutions for problem (1.2) with the exponents satisfying $1 < q < p$. More relevant information about quasilinear elliptic problems may be refereed to [11–15] and the references therein. In particular, the authors of [13] and [15] discussed the existence of multiple positive solutions for p-Laplacian elliptic equations when the weight functions f and g are continuous and satisfy some additional conditions.

Recently, the existence of solutions for elliptic systems were studied extensively, the readers may refer to [16–22]. Hsu and Li [23] concerned about the following semilinear elliptic system with concave-convex nonlinearities:

$$\begin{cases} -\Delta u - t\dfrac{u}{|x|^2} = \lambda|u|^{q-2}u + \dfrac{2\alpha}{\alpha+\beta}\dfrac{|u|^{\alpha-2}u|v|^\beta}{|x|^s} & \text{in } \Omega\setminus\{0\}, \\ -\Delta v - t\dfrac{v}{|x|^2} = \mu|v|^{q-2}v + \dfrac{2\beta}{\alpha+\beta}\dfrac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^s} & \text{in } \Omega\setminus\{0\}, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\lambda, \mu > 0, 0 \leq t < \bar{t} = (\frac{N-2}{2})^2, 1 \leq q < 2, 0 \leq s < 2$, and $\alpha,\ \beta > 1$ satisfy $\alpha + \beta = 2^*(s) = \frac{2(N-s)}{N-2}$. They obtained the existence of multiple positive solutions by means of variational methods when the parameters λ, μ satisfy $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda$, where Λ is a positive constant.

However, as far as we know, there are few results of the quasilinear elliptic system with critical Sobolev-Hardy exponents and nonlinearities. Motivated by the results of above papers, in this paper, we discuss the existence of multiple solutions to problem (1.1) by a variational method involving the Nehari manifold which is similar to the fibrering method (see [24, 25]). The first positive solution can be obtained by using the same argument as that in the subcritical case. In order to obtain the existence of the second positive solution, we have to add restrictions on the functions f, g, h to prove the compactness of the extraction of the Palais-Smale sequences in the Nehari manifold.

Throughout this paper, we make some assumptions on the weight functions f, g and h as the following:

(H0) $1 < q < p,\ \lambda,\ \mu > 0,\ 0 \leq l < \bar{l},\ 0 \leq s,\ t < p,\ \alpha > 1,\ \beta > 1$ and $\alpha + \beta = p^*(t)$;

(H1) $f, g \in C(\overline{\Omega}),\ f(x),\ g(x) \geq 0$ and $f(x),\ g(x) \not\equiv 0$ in Ω;

(H2) there exist β_0 and $\rho_0 > 0$ such that $B_{2\rho_0}(0) \subset \Omega$ and $f(x),\ g(x) \geq \beta_0$ for all $x \in B_{2\rho_0}(0) \subset \Omega$;