Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2013,**33B**(1):218–226
Acta Mathematica Scientia 数学物理学报
http://actams.wipm.ac.cn

OSCILLATORY BEHAVIOR OF SOLUTIONS OF CERTAIN THIRD ORDER MIXED NEUTRAL DIFFERENCE EQUATIONS*

E. THANDAPANI   N. KAVITHA
Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai 600 005, India
E-mail: ethandapani@yahoo.co.in; kavitha_snd@hotmail.com

Abstract The objective of this paper is to study the oscillatory and asymptotic properties of the mixed type third order neutral difference equation of the form

$$\Delta\left(a_n\Delta^2\left(x_n+b_nx_{n-\tau_1}+c_nx_{n+\tau_2}\right)\right)+q_nx^{\beta}_{n+1-\sigma_1}+p_nx^{\beta}_{n+1+\sigma_2}=0,$$

where $\{a_n\},\{b_n\},\{c_n\},\{q_n\}$ and $\{p_n\}$ are positive real sequences, β is a ratio of odd positive integers, $\tau_1,\ \tau_2,\ \sigma_1$ and σ_2 are positive integers. We establish some sufficient conditions which ensure that all solutions are either oscillatory or converges to zero. Some examples are presented to illustrate the main results.

Key words oscillation; third order; mixed type neutral difference equation

2010 MR Subject Classification 39A10

1 Introduction

In this paper, we are concerned with the following third order mixed type neutral difference equation of the form

$$\Delta\left(a_n\Delta^2\left(x_n+b_nx_{n-\tau_1}+c_nx_{n+\tau_2}\right)\right)+q_nx^{\beta}_{n+1-\sigma_1}+p_nx^{\beta}_{n+1+\sigma_2}=0,\tag{1.1}$$

where $n\in\mathbb{N}=\{n_0,n_0+1,\cdots\},n_0$ is a nonnegative integer. We assume the following assumptions to hold:

(H$_1$) $\{a_n\}$ is a positive nondecreasing sequence such that $\sum\limits_{n=n_0}^{\infty}\dfrac{1}{a_n}=\infty$;

(H$_2$) $\{b_n\}$ and $\{c_n\}$ are real sequences such that $0\le b_n\le b$ and $0\le c_n\le c$ with $b+c<1$;

(H$_3$) $\{p_n\}$ and $\{q_n\}$ are positive real sequences;

(H$_4$) β is a ratio of odd positive integers, $\tau_1,\ \tau_2,\ \sigma_1$ and σ_2 are nonnegative integers.

Let $\theta=\max\{\tau_1,\sigma_1\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n\ge n_0-\theta$ and satisfying equation (1.1) for all $n\in\mathbb{N}$. A nontrivial solution

*Received September 2, 2011; revised February 27, 2012.

$\{x_n\}$ is said to be nonoscillatory if it is either eventually positive or eventually negative and it is oscillatory otherwise.

Recently there was increasing interest in studying the oscillatory behavior of neutral type difference equation, see for example [1–16] and the references cited therein. This is due to the fact that such equations have various applications in problems dealing with vibrating masses attached to an elastic bar and in some variational problems.

Regarding the mixed type neutral difference equations, Grace [8] considered the third order, mixed type neutral difference equation

$$\Delta^3 (x_n + a\ x_{n-m} - b\ x_{n+k}) \pm (q\ x_{n-g} + p\ x_{n+h}) = 0 \tag{1.2}$$

and established some sufficient conditions for the oscillation of all solutions of equation (1.2).

Agarwal, Grace and Bohner considered the m-th order mixed neutral difference equation

$$\Delta^m (x_n + a\ x_{n-k} + b\ x_{n+\sigma}) + q\ x_{n-g} + p\ x_{n+h} = 0 \tag{1.3}$$

and obtained some oscillation theorems for the oscillation of all solutions of equation (1.3).

In [6], Agarwal and Grace considered several third order mixed neutral difference equations and established some sufficient conditions for the oscillation of all solutions. It is to be noted that all the results obtained only for linear type equations and to the best of our knowledge, there is no paper dealing with nonlinear mixed type neutral difference equations. Therefore, our aim is to study the oscillatory and asymptotic behavior of solutions of equation (1.1).

The paper is organized as follows: In Section 2, we present some sufficient conditions which ensure that every solution $\{x_n\}$ of equation (1.1) is either oscillatory or $x_n \to 0$ as $n \to \infty$. In Section 3, examples are presented to illustrate the main results.

2 Oscillation Results

In this section, we present some new oscillation criteria for equation (1.1). For the sake of convenience, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large n.

We begin with the following lemmas which are crucial in the proof of the main results. For simplicity, we use the following notations:

$$z_n = x_n + b_n x_{n-\tau_1} + c_n x_{n+\tau_2}, \quad R_n = Q_n + P_n,$$

$$Q_n = \min\{q_n, q_{n-\tau_1}, q_{n+\tau_2}\}, \quad P_n = \min\{p_n, p_{n-\tau_1}, p_{n+\tau_2}\},$$

$$\eta_n = \left(\frac{d}{4}\right)^{\beta-1} \frac{k\,(n-\sigma_1)^\beta}{2^\beta} R_n \quad \text{for some } k \in (0,1) \text{ and } d > 0.$$

Lemma 2.1 Assume $A \geq 0,\ B \geq 0,\ \beta \geq 1$. Then

$$(A+B)^\beta \leq 2^{\beta-1}(A^\beta + B^\beta).$$

Proof The proof of the lemma is simple and so omitted. □

Lemma 2.2 Let $\{x_n\}$ be a positive solution of equation (1.1). Then there are only two cases for $n \geq n_1 \in \mathbb{N}$ sufficiently large: