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INTERPOLATION SOLUTION TO A BOUNDARY VALUE PROBLEM OF HARMONIC FIELD*

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Abstract This article is a improvement on author's early work (Acta Mathematica Scientia, Vol.30 No.2 Ser.A 2010). In this article, there are two new contributions: 1) The restrictive conditions on approximation domain boundary is improved essentially. 2) The Fejér points is extended by perturbed Fejér points with stable order of approximation.

Key words Harmonic interpolation approximation; perturbed Fejér points; stable order of approximate; boundary restriction condition

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1 Introduction

Suppose that D is a simply–connected domain bounded by a closed Jordan curve Γ, $z = 0 \in D$. If function $u(x,y)$ has continuous second partial derivatives satisfying Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ (x,y) \in D,$$

it is well–known that $u(x,y)$ is called harmonic function in D.

Let $g_1(x,y)$ and $g_2(x,y)$ be continuous functions on Γ. Whether there exists in D a harmonic function $u(x,y)$ with partial derivatives satisfying

$$\frac{\partial u}{\partial x} = g_1(x,y), \quad \frac{\partial u}{\partial y} = g_2(x,y), \quad (x,y) \in \Gamma. \tag{1.1}$$

For this problem, the authors have proved that [1]:

(I) The sufficient and necessary conditions for the existence of such a harmonic function, that is, that $g_1(x,y)$ and $g_2(x,y)$ satisfy the following conditions:

1° The continuity domain of both $g_1(x,y)$ and $g_2(x,y)$ can be extended to $\overline{D}$.

2° $g_1(x,y)$ and $g_2(x,y)$ have continuous partial derivatives in D and satisfying

$$\frac{\partial g_1}{\partial y} = \frac{\partial g_2}{\partial x}, \quad \frac{\partial g_1}{\partial x} = -\frac{\partial g_2}{\partial y}, \quad (x,y) \in D. \tag{1.2}$$

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Consequently, there is

Definition 1.1 If $u(x,y)$ is harmonic in D and with $\frac{\partial u}{\partial x}=g_1(x,y)$ and $\frac{\partial u}{\partial y}=g_2(x,y)$ which are continuous and satisfy 1° and 2°, then it is said to be $u(x,y)\in H^1(\overline{D})$.

Moreover, if $u(x,y)\in H^1(\overline{D})$, then $u(x,y)$ can be represented as following linear integration

$$u(x,y)=\int_{(0,0)}^{(x,y)} g_1(x,y)\mathrm{d}x+g_2(x,y)\mathrm{d}y,\quad (x,y)\in\overline{D}. \tag{1.3}$$

(II) For any $u(x,y)\in H^1(\overline{D})$,there exists a analytic function

$$f(x,y)=\int_{(0,0)}^{(x,y)} g_1(x,y)\mathrm{d}x+g_2(x,y)\mathrm{d}y+\mathrm{i}\int_{(0,0)}^{(x,y)} -g_2(x,y)\mathrm{d}x+g_1(x,y)\mathrm{d}y \tag{1.4}$$

with $\mathrm{Re}\{f(z)\}=u(x,y)$ and continuous derivation on $\overline{D}$,

$$f'(z)=g_1(x,y)+\frac{1}{\mathrm{i}}g_2(x,y).$$

(III) By the help of complex Lagrange interpolation approximation to $f\in A^1(\overline{D})$ (that is, $f(z)$ analytic in D and $f'(z)$ continuous on $\overline{D}$), authors proved that suppose $\Gamma\in J_0$, there exists a sequence of interpolation harmonic polynomial $\{u_n(x,y)\}$ at Fejér points which uniformly approximates to $u(x,y)\in H^1(\overline{D})$ with the order of $O\big(\frac{\ln n}{n}\omega(f',\frac{1}{n})\big)$.

What does the symbol J_0 mean? We say $\Gamma\in J_0$, if Γ is a smooth Jordan closed curve, $z=\psi(w)$ is the exterior conformal mapping (see Section 3), and the modulus of continuity of $\arg\psi'(w)$ on $|w|=1$ is denoted by $\lambda(t)$ which satisfies $\int_0^c\frac{\lambda(t)}{t}|\ln t|\mathrm{d}t<+\infty$, $a>0$ [2, p.632].

As is well known, the behavior of interpolation is dependent on the interpolation nodes in realty. Hence, the research on the perturbation of nodes is interested in practice and theory.

In this article, the stable order of approximation by harmonic interpolation at perturbed Fejér points is obtained.

At the same time, the authors will try best to decrease the restrictive condition on $\Gamma=\partial D$ as far as possible, so as to add the applicable extensiveness of harmonic interpolation approximation.

2 Perturbed Roots of Unity

Definition 2.1 $\{w_k^*=\mathrm{e}^{\frac{2k+t_k}{n}\pi\mathrm{i}}\}_0^{n-1}$ are called perturbed roots of unity, in which $\sum\limits_{k=0}^{n-1}|t_k|=t$, $0\le t<\frac{4}{\pi+2}$, and $t_n=t_0$.

Clearly, $w_k^*\equiv w_k$ as $t=0$.

Note that for the different problem, the perturbations may be different [3, 4] each other.

Let

$$\omega_n^*(w)=\prod_{k=0}^{n-1}(w-w_k^*),$$

$$l_k^*(w)=\prod_{\substack{l=0\\l\neq k}}^{n-1}\frac{w-w_l^*}{w_k^*-w_l^*}=\frac{\omega_n^*(w)}{\omega_n^{*\prime}(w_k^*)(w-w_k^*)},$$

$$L_n^*(f,w)=\sum_{k=0}^{n-1}l_k^*(w)f(w_k^*).$$