Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2013,**33B**(2):484–494
Acta Mathematica Scientia
数学物理学报
http://actams.wipm.ac.cn

ON A NONLINEAR VOLTERRA-HAMMERSTEIN INTEGRAL EQUATION IN TWO VARIABLES*

Le Thi Phuong Ngoc
Nhatrang Educational College, 01 Nguyen Chanh Str., Nhatrang City, Vietnam
E-mail: *ngoc1966@gmail.com; ngocltp@gmail.com*

Nguyen Thanh Long
Department of Mathematics and Computer Science, University of Natural Science, Vietnam National University Ho Chi Minh City, 227 Nguyen Van Cu Str., Dist. 5, Ho Chi Minh City, Vietnam
E-mail: *longnt2@gmail.com; longnt1@yahoo.com*

Abstract Using a fixed point theorem of Krasnosel'skii type, this article proves the existence of asymptotically stable solutions for a Volterra-Hammerstein integral equation in two variables.

Key words Volterra-Hammerstein integral equation in two variables; the fixed point theorem of Krasnosel'skii type; contraction mapping; completely continuous; asymptotically stable solution

2010 MR Subject Classification 47H10; 45G10; 47N20; 65J15

1 Introduction

In this article, we consider the nonlinear Volterra-Hammerstein integral equation in two variables of the form

$$\begin{aligned} u(x,y) = q(x,y) + f(x,y,u(x,y)) + \int_0^x \int_0^y V\left(x,y,s,t,u(s,t)\right) \mathrm{d}s\mathrm{d}t \\ + \int_0^\infty \int_0^\infty F\left(x,y,s,t,u(s,t)\right) \mathrm{d}s\mathrm{d}t, \end{aligned} \tag{1.1}$$

$(x,y) \in \mathbb{R}_+^2$, where $\mathbb{R}_+ = [0,\infty)$, $q : \mathbb{R}_+^2 \to E$; $f : \mathbb{R}_+^2 \times E \to E$; $F : \mathbb{R}_+^4 \times E \to E$; $V : \Delta \times E \to E$ are supposed to be continuous, $\Delta = \{(x,y,s,t) \in \mathbb{R}_+^4 : s \le x,\ t \le y\}$, and E is a Banach space with norm $|\cdot|$.

The integral equations in a variable or two variables have been extensively studied by many authors, for example, we refer to [1–8] and the references given therein. In general, existence theorems for these equations are proved by the use of the fundamental methods in which the fixed point theorems are often applied. In [1, 2], Avramescu and Vladimirescu have proved the existence of asymptotically stable solutions to the following integral equations

$$u(t) = q(t) + f(t,u(t)) + \int_0^t V(t,s)u(s)\mathrm{d}s + \int_0^t G(t,s,u(s))\mathrm{d}s,\ \ t \in \mathbb{R}_+, \tag{1.2}$$

*Received July 11, 2011. The authors are extremely grateful for the support given by Vietnam's National Foundation for Science and Technology Development (NAFOSTED) under Project 101.01-2012.12.

or

$$u(t) = q(t) + \int_0^t K(t, s, u(s))\mathrm{d}s + \int_0^\infty G(t, s, u(s))\mathrm{d}s, \ t \in \mathbb{R}_+, \tag{1.3}$$

under suitable hypotheses. In the proofs, a fixed point theorem of Krasnosel'skii type is used, (see [1, 2]).

In case the Banach space E is arbitrary, recently in [6, 7], the existence of asymptotically stable solutions to the following integral equations

$$x(t) = q(t) + f(t, x(t)) + \int_0^t V(t, s, x(s))\mathrm{d}s + \int_0^t G(t, s, x(s))\mathrm{d}s, \ t \in \mathbb{R}_+, \tag{1.4}$$

or

$$x(t) = q(t) + f(t, x(t)) + \int_0^t V(t, s, x(s))\mathrm{d}s + \int_0^\infty G(t, s, x(s))\mathrm{d}s, \ t \in \mathbb{R}_+, \tag{1.5}$$

also have been proved using the fixed point theorem of Krasnosel'skii type as follows.

Theorem 1.1 Let $(X, |\cdot|_n)$ be a Fréchet space and let U, $C : X \to X$ be two operators. Assume that

(i) U is a k-contraction operator, $k \in (0, 1)$ (depending on n), with respect to a family of seminorms $\|\cdot\|_n$ equivalent with the family $|\cdot|_n$;

(ii) C is completely continuous;

(iii) $\lim\limits_{|x|_n \to \infty} \frac{|Cx|_n}{|x|_n} = 0$, $\forall n \in \mathbb{N}$. Then, $U + C$ has a fixed point.

In [5], Lungu and Rus established some results relative to existence, uniqueness, integral inequalities, and data dependence for the solutions of the following functional Volterra-Fredholm integral equation with deviating argument in a Banach space by Picard operators technique

$$u(x, y) = g(x, y, h(u)(x, y)) + \int_0^x \int_0^y K\left(x, y, s, t, u(s, t)\right)\mathrm{d}s\mathrm{d}t, \ x, \ y \in \mathbb{R}_+. \tag{1.6}$$

In [8], on the basis of the applications of the Banach fixed point theorem coupled with Bielecki type norm and the integral inequality with explicit estimates, B. G. Pachpatte studied some basic properties of solutions of the Fredholm type integral equation in two variables as follows

$$u(x, y) = f(x, y) + \int_0^a \int_0^b g\left(x, y, s, t, u(s, t), D_1 u(s, t), D_2 u(s, t)\right)\mathrm{d}t\mathrm{d}s. \tag{1.7}$$

With the same methods, in [9], the existence, uniqueness, and other properties of solutions of certain Volterra integral and integrodifferential equations in two variables were considered.

The functional integral equations in two variables mentioned as above lead to the study of the existence of asymptotically stable solutions of (1.1).

Also applying Theorem 1.1, with adding some suitable conditions, we also obtain the same results for (1.1) as those for (1.5) in [7]. The proofs are completed by combination of the arguments in [7] with appropriate modifications and the integral inequalities with explicit estimates (see Lemma 2.2 and Lemma 3.2 below).

The article consists of three sections and the existence of solutions, the existence of asymptotically stable solutions for (1.1) will be presented in Sections 2, 3. The results obtained here may be considered as the generalizations of those in [7].