Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2013,**33B**(2):525–531
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

CHARACTERIZATION OF MODULAR FROBENIUS GROUPS OF SPECIAL TYPE*

Juanjuan FAN (范娟娟)
Department of Economics and Trade, Fujian Jiangxia University, Fuzhou 350108, China
E-mail: *mayga@163.com*

Ni DU (杜妮)† *Jiwen ZENG* (曾吉文)
School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
E-mail: *duni@xmu.edu.cn; jwzeng@xmu.edu.cn*

Abstract In this article, we first investigate the properties of modular Frobenius groups. Then, we consider the case that G' is a minimal normal subgroup of a modular Frobenius group G. We give the complete classification of G when G' as a modular Frobenius kernel has no more than four conjugacy classes in G.

Key words Modular Frobenius group; minimal normal subgroup; Frobenius group; conjugacy classes

2010 MR Subject Classification 20C15; 20C20

1 Introduction

Let G be a finite group and fix a prime number p. The following definition of p-modular Frobenius group was first introduced by Kuisch and Waall in [1].

Definition 1.1 Let $1 < N \lhd G$. Suppose that K is a splitting field for $K[N]$ with char(K)= $p > 0$. We say G is a p-modular Frobenius group if G satisfies one of the following equivalent conditions.

(a) Every nontrivial irreducible $K[N]$-module V has the property that V^G is irreducible.

(b) $C_G(x) \leq N$ for every nontrivial p-regular element x of N.

Such a normal subgroup N, as occurring in Definition 1.1, is called a p-modular Frobenius kernel. It is obvious that G is an ordinary Frobenius group with kernel N if either $|G|$ or $|N|$ is coprime to p by Definition 1.1. It is easy to see that a normal p-subgroup is a trivial p-modular Frobenius kernkel. In this article, we deal with p-modular Frobenius kernkels that are not p-subgroups.

From now on, we may assume that $K = F$ is an algebraically closed field of characteristic p at the given fixed prime p. We write IBr(G) for the set of irreducible Brauer characters of

*Received February 25, 2011; revised October 20, 2012. Supported by the National Natural Science Foundation of China (11171243, 11201385), the Technology Project of Department of Education of Fujian Province (JA12336), the Fundamental Research Funds for the Central Universities (2010121003), and the Science and the Natural Science Foundation of Fujian Province (2011J01022).

†Corresponding author: Ni DU.

G. FG-modules are finitely generated right FG-modules. Via the correspondence between FG-modules and FG-representations, we can obtain another equivalent definition of p-modular Frobenius group as the following:

Definition 1.2 We say that G is a p-modular Frobenius group with kernel N if every non-principal irreducible Brauer character of N induces irreducibly to G.

It holds that a Frobenius group G is a p-modular Frobenius group when its Frobenius kernel is not a p-group. Conversely, it might not be correct that a p-modular Frobenius group is an ordinary Frobenius group. Moreover, a p-modular Frobenius kernel is not necessarily nilpotent. A counterexample is $\mathrm{GL}(2,3)$. $\mathrm{GL}(2,3)$ is a 2-modular Frobenius group with kernel $\mathrm{SL}(2,3)$ which is 3-nilpotent but not 2-nilpotent, while $\mathrm{GL}(2,3)$ is not a Frobenius group.

In Section 2, we give some basic properties of p-modular Frobenius groups.

In Section 3, we obtain the complete classification of p-modular Frobenius groups according to the numbers of conjugacy classes contained in their derived groups, where the derived groups being modular Frobenius kernels are the minimal normal subgroups.

Now, we give the reason why we consider the case that a p-modular Frobenius kernel is a minimal normal subgroup. Let G be a p-modular Frobenius group with kernel N and H be a minimal normal subgroup of G. Then $H \leq N$ which follows by Lemma 2.3(d). If $H < N$, it follows that G/H is a p-modular Frobenius group with kernel N/H by Lemma 2.3(b). Suppose that H_1/H is a minimal normal subgroup of G/H. Then, it follows that H_1/H is contained in N/H by Lemma 2.3(d). If $H_1/H < N/H$, then G/H_1 is a p-modular Frobenius group with kernel N/H_1 by Lemma 2.3(b). We can repeat the process and find $M \lhd G$ with $M < N$ such that G/M is a p-modular Frobenius group with kernel N/M, where N/M is the minimal normal subgroup of G/M. So, we consider the case that a p-modular Frobenius kernel is a minimal normal subgroup of G. By Lemma 3.1(b), we also consider the case that the derived group as a modular Frobenius kernel is a minimal normal subgroup. It is well known that a normal subgroup of G is the union of some conjugacy classes in G. And a minimal normal subgroup of G might have few conjugacy classes in G. Let G be a p-modular Frobenius group with kernel G' and G' be a minimal normal subgroup of G. Inspired by Riese and Shahabi [2], and Shahryari and Shahabi in [3, 4], we obtain the structure of G as the following.

Theorem 1.3 Let G be a p-modular Frobenius group with kernel G'. Suppose that G' is the minimal normal subgroup of G.

(a) If G' is the union of two conjugacy classes in G, then G is a doubly transitive Frobenius group $(Z_r)^s \rtimes Z_{r^s-1}$ for some prime $r \neq p$. In addition, $|\mathrm{Irr}_1(G)| = 1$.

(b) If G' is the union of three conjugacy classes in G, then G is a Frobenius group of the kind $(Z_r)^s \rtimes Z_{\frac{r^s-1}{2}}$ for some odd prime $r \neq p$ and $s \geq 2$. In addition, $|\mathrm{Irr}_1(G)| = 2$.

(c) If G' is the union of four conjugacy classes in G, then G is a Frobenius group $(Z_r)^s \rtimes Z_{\frac{r^s-1}{3}}$ for some prime $r \neq p$. In addition, $|\mathrm{Irr}_1(G)| = 3$.

Before stating one of the results, we introduce some notations. If $N \lhd G$ and $\alpha \in \mathrm{IBr}(N)$, we write $\mathrm{IBr}(\alpha^G) = \{\varphi \in \mathrm{IBr}(G) \mid \mathrm{I}(\alpha^G, \varphi) \neq 0\}$. Let $\mathrm{IBr}(G)^\sharp$ denote the set of non-principal irreducible Brauer characters of G and $N^\sharp$ denote $N - \{1\}$. We use the notation $\mathrm{LBr}(G)$ to denote the set of Brauer linear characters of G. Of course, Brauer linear characters are irreducible. We write $(Z_r)^n$ and Z_m to denote an elementary abelian group of order r^n and a