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EXISTENCE OF MEROMORPHIC SOLUTIONS OF SOME HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS*

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Abstract This article discusses the problems on the existence of meromorphic solutions of some higher order linear differential equations with meromorphic coefficients. Some nice results are obtained. And these results perfect the complex oscillation theory of meromorphic solutions of linear differential equations.

Key words Higher order linear differential equation; meromorphic coefficient; meromorphic solutions; the first coefficient

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1 Introduction

Many results were obtained about the properties of meromorphic solutions of linear differential equations with meromorphic coefficients. However, there were few studies on the existence of meromorphic solutions of such equations. Therefore, it is necessary to investigate the existence of meromorphic solutions of linear differential equations.

As known, a linear differential equation with holomorphic coefficients must have holomorphic solutions. But the characteristic of solutions is more complicated for a linear differential equation with meromorphic coefficients, as found in [1, Corollary 6.9], Laine obtained

Theorem A Let $G \subset C$ be a simply connected domain such that $A(z)$ is meromorphic in G. The differential equation

$$f'' + A(z)f = 0$$

admits two linearly independent meromorphic solutions in G if and only if, at all poles z_0 of $A(z)$, there are some of them, and the Laurent expansion of $A(z)$ is of the form

$$A(z) = \frac{b_0}{(z-z_0)^2} + \frac{b_1}{z-z_0} + b_2 + \cdots ,$$

where

$$4b_0 = 1 - m^2, m \text{ is an odd integer} \geq 3.$$

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Our major objective in this article is to discuss the existence of meromorphic solutions of some higher order linear differential equations with meromorphic coefficients using a method different from that in [1].

First, we recall the definition of meromorphic function.

Definition 1 Let $M(z)$ be a meromorphic function. Then, $M(z)$ can be represented by Laurent series

$$M(z) = \sum_{n=-t}^{\infty} m_n (z-b)^n \quad (m_{-t} \neq 0)$$

for $z \in D(b,\epsilon) = \{z, |z-b| < \epsilon\}$, where m_{-t} is the first coefficient of the expansion.

For convenience, we can denote the region $D(b,\epsilon)$ by $D(0,\epsilon)$. Throughout this article, all the holomorphic functions are holomorphic in $D(0,\epsilon)$. So are the meromorphic functions.

In this article, we mainly consider such type of higher order linear differential equations as

$$f^{(k)}(z) - A(z)f(z) = 0, \tag{1}$$

where k is a positive integer, and $A(z) = \sum\limits_{n=-t}^{\infty} x_n z^n (x_{-t} \neq 0)$ is a meromorphic function in $D(0,\epsilon)$. The equation (1) is also called an k-order equation with meromorphic coefficient having a t-order pole.

It is evident that equation (1) is equivalent to

$$z^t f^{(k)}(z) - H(z)f(z) = 0, \tag{2}$$

where $H(z) = \sum\limits_{n=0}^{\infty} h_n z^n = z^t A(z) = \sum\limits_{n=-t}^{\infty} x_n z^{n+t}$ $(h_0 = x_{-t} \neq 0)$ for $z \in D(0,\epsilon)$ and t is a positive integer.

The equation

$$h_0 = x(x-1)\cdots(x-k+1) \tag{3}$$

is called the first coefficient equation.

In what follows, we shall verify the following three theorems.

Theorem 1 Let $H(z) = \sum\limits_{n=0}^{\infty} h_n z^n$ be a holomorphic function in $D(0,\epsilon)$ and $H(0) = h_0 \neq 0$. Then, the equation

$$z^i f^{(k)}(z) - H(z)f(z) = 0 \tag{4}$$

has and only has $k-i$ linearly independent holomorphic solutions of the form $f_j(z) = \sum\limits_{n=j}^{\infty} a_j z^j$ and $a_j \neq 0$, where $j = i, i+1, \cdots, k-1, i \in \{1, 2, \cdots, k-1\}$.

Theorem 2 Let $H(z) = \sum\limits_{n=0}^{\infty} h_n z^n$ be a holomorphic function in $D(0,\epsilon)$ and $H(0) = h_0 \neq 0$. If equation (3) has v distinct positive integer solutions $p_1, p_2, \cdots, p_v$, then, the equation

$$z^k f^{(k)}(z) - H(z)f(z) = 0 \tag{5}$$

has v linearly independent holomorphic solutions of the form $f(z) = \sum\limits_{n=p_j}^{\infty} a_n z^n$ and $a_{p_j} \neq 0$, where $p_j \geq k, j = 1, 2, \cdots, v$.

Theorem 3 Let $H(z) = \sum\limits_{n=0}^{\infty} h_n z^n$ be a holomorphic function in $D(0,\epsilon)$ and $H(0) = h_0 \neq 0$. Assume that equation (3) has a negative integer solution $-l$, then for equation (5), we have