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A SUFFICIENT CONDITION OF CONVERGENCE FOR CLIFFORD CONTINUED FRACTIONS*

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Abstract In this article, a sufficient condition for a Clifford continued fraction to be convergent is established, and some applications are given.

Key words Clifford continued fraction; sufficient condition; convergence; application

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1 Preliminaries

Let A_m denote the associative algebra over the reals $\mathbb{R}$ generated by 1, e_1, e_2, $\cdots$, e_{m-1} subject to the relations $e_i^2 = -1$ and $e_ie_j = -e_je_i$ $(i \neq j)$, $i, j = 1, 2, \cdots, m-1$ [12]. Each element $a \in A_m$ has a unique representation in the form

$$a = a_0 + \sum a_v E_v, \tag{1.1}$$

where a_0, a_v are reals, the summation is over all multi-indices $v = (v_1, v_2, \cdots, v_p)$ with $0 < v_1 < v_2 < \cdots < v_p \le m-1$ and $E_v = e_{v_1}e_{v_2}\cdots e_{v_p}$. a_0 is said to be the real part of a, denoted by $a_0 = \mathrm{Re}(a)$. The modulus of a is defined by

$$|a| = (a_0^2 + \sum a_v^2)^{\frac{1}{2}}.$$

Let a' be the element obtained from a by replacing each e_i in (1.1) by $-e_i$, a^* the element obtained from a by reversing the order of the factors in each $E_v = e_{v_1}e_{v_2}\cdots e_{v_p}$, and $\overline{a} = (a^*)' = (a')^*$. Obviously, $(a+b)' = a' + b'$, $(ab)' = a'b'$ and $(ab)^* = b^*a^*$.

All the elements $x = x_0 + x_1e_1 + \cdots + x_{m-1}e_{m-1}$ $(x_k \in \mathbb{R}$, $k = 0, 1, \cdots, m-1)$ are said to be vectorial elements in A_m, denoted by $x \in \mathbb{R}^m$. Let Γ_m be the set of all elements in A_m which can be expressed as a finite product of non-zero vectorial elements of A_m. It is said to be the m-dimensional Clifford group. For any a, $b \in \Gamma_m$, we know that $|a|^2 = a\overline{a} = \overline{a}a$ and

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$|ab| = |ba| = |a||b|$ [1–3], and for any $a \in \Gamma_m$, a is invertible. In the following, we assume that $a^{-1} = \infty$ if $a = 0$ and $a^{-1} = 0$ if $a = \infty$.

Definition 1 $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is said to be an m-dimensional Clifford matrix if

i) $a, b, c, d \in \Gamma_m \bigcup \{0\}$;

ii) $\Delta(A) = ad^* - bc^* \in \mathbb{R} \setminus \{0\}$;

iii) $ab^*, cd^*, a^*c, b^*d \in \mathbb{R}^m$.

Let $GL(2, \Gamma_m)$ denote the group of all m-dimensional Clifford matrices with the matrix product operation. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \Gamma_m)$ correspond to the mapping in $\overline{\mathbb{R}}^m$:

$$s: \quad x \mapsto sx = (ax + b)(cx + d)^{-1}.$$

Then, s is a bijective mapping from $\overline{\mathbb{R}}^m$ onto itself [2]. In the following, we always write $s = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. And we call $\overline{s} = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$ the conjugate transpose matrix of s.

As in [9, 10], we give the following definition:

Definition 2 A Clifford continued fraction is an ordered pair $(\{b_n\}, \{c_n\}, \{d_n\}; \{f_n\})$, where $\{b_n\}_1^\infty, \{c_n\}_1^\infty$, and $\{d_n\}_1^\infty$ are given sequences of Clifford numbers in $\Gamma_m \bigcup \{0\}$ with $s_n = \begin{pmatrix} 0 & b_n \\ c_n & d_n \end{pmatrix} \in GL(2, \Gamma_m)$ $(n \geq 1)$, and $\{f_n\}_1^\infty$ is the sequence in $\overline{\mathbb{R}}^m$ defined by

$$f_n = S_n(0) \quad (n \geq 1),$$

where

$$S_n = s_1 \circ s_2 \circ \cdots \circ s_n.$$

For convenience, in general, we denote a Clifford continued fraction $(\{b_n\}, \{c_n\}, \{d_n\}; \{f_n\})$ by the symbol

$$\frac{b_1}{d_1} + \frac{c_1 b_2}{d_2} + \frac{c_2 b_3}{d_3} + \cdots. \tag{1.2}$$

Then, we call $\{f_n\}$ the sequence of approximants of (1.2) and f_n the nth approximant.

Definition 3 Clifford continued fractions $(\{b_n\}, \{c_n\}, \{d_n\}; \{f_n\})$ and $(\{b_n^*\}, \{c_n^*\}, \{d_n^*\}; \{f_n^*\})$ are said to be equivalent if $f_n = f_n^*$ for all n.

A Clifford continued fraction (1.2) is said to be convergent if its sequence of approximants $\{f_n\}$ converges to a point in $\overline{\mathbb{R}}^m$.

In complex continued fractions, the following question is interesting.

Question 1 When does a continued fraction converge?

About Question 1, the following were proved.

Theorem A [7, 11] Let $K(1/b_n)$ be a continued fraction with positive elements b_n.

i) If f_n denotes the nth approximant, then,

$$f_{2n-1} > f_{2n+1} > f_{2n+2} > f_{2n}, \qquad n = 1, 2, 3 \cdots,$$

so that the even and odd parts of $K(1/b_n)$ both converge to finite values.

ii) If, in addition, $\sum b_n = \infty$, then, the continued fraction converges to a finite value f and

$$|f - f_n| < |f_n - f_{n-1}|, \qquad n = 2, 3, 4, \cdots.$$