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L^2-ERROR OF EXTRAPOLATION CASCADIC MULTIGRID (EXCMG)*

Dedicated to Professor Wu Wenjun on the occasion of his 90th birthday

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Abstract Based on an asymptotic expansion of finite element, an extrapolation cascadic multigrid method (EXCMG) is proposed, in which the new extrapolation and quadratic interpolation are used to provide a better initial value on refined grid. In the case of multiple grids, both superconvergence error in H^1-norm and the optimal error in l^2-norm are analyzed. The numerical experiment shows the advantage of EXCMG in comparison with CMG.

Key words cascadic multigrid; finite element; new extrapolation; quadratic interpolation; L^2-error

2000 MR Subject Classification 65N30

1 Introduction

To solve a linear system of equations derived by the finite difference method or finite element method, people's aim is to get the solution of N-order system by $O(N)$ the operations. Multigrid method (MG) has first realized this purpose and then becomes the most effective algorithm to solve the large scale problem. Since 1970's, MG has been extensively studied and the systematical theory has been established. There are two important types: (Classical) Multigrid Method (MG) was proposed early, see [1, 2]. Three operators between different grids, e.g. the interpolation, restriction and iteration, are used. Its code looks complicated. Cascadic Multigrid Method (CMG) was proposed by Borneman, Deunfhard [3], and then analyzed by Z.C. Shi, X.J. Xu [4–7]. In which only the interpolation and iteration from coarse grids to refined grids are used, its code is easily realized and more remarkable.

Consider an elliptic problem in a planar polygon Ω with the boundary Γ

$$-\Delta u = f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \ \Gamma.$$

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Denote by $u \in H_0^1 = \{u : \ u \in H^1(\Omega), \ u = 0 \text{ on } \Gamma\}$ the exact solution of the problem. Assume that the domain Ω is subdivided into a sequence of grids $Z_l, l = 0, 1, 2, \cdots, L$ with step-length $h_l = h_0/2^l$. Denote by $V_l \subset H_0^1$ a linear (or bilinear) finite element subspace on the grid Z_l and by $U^l \in V_l$ the corresponding finite element solution satisfying

$$A(U^l, v) = (f, v), \quad v \in V_l, \quad l = 0, 1, 2, \cdots, L,$$

where the bilinear form $A(u, v) = \int_\Omega \nabla u \nabla v \mathrm{d}x$ is bounded and H_0^1-coercive. This problem is reduced to a linear system of equations $K_l U^l = f^l$ with the nodal unknowns U^l of dimension n_l. Taking any initial value $U^{l,0}$ and using an iterative operator S_l, the system can be solved by an iteration process, $U^{l,k+1} = S_l U^{l,k} + F^l, \quad k = 0, 1, 2, \cdots, k_l$. The final aim is to get the iterative solution U^{L,k_L} on the finest grid Z_L.

Because the exact finite element U^l also satisfies $U^l = S_l U^l + F^l$, the iteration error $e^{l,k} = U^{l,k} - U^l \in V_l$ satisfies iteration relation $e^{l,k+1} = S_l e^{l,k} = S_l^k e^{l,0}$. Assume that $\{\rho_j, \phi_j\}_{j=1}^{n_l}$ are the eigen-system of the operator S_l and the initial error can be expressed by $e^{l,0} = \sum\limits_{j=1}^{n_l} a_j \phi_j, \|e^{l,0}\|^2 = \sum\limits_{j=1}^{n_l} |a_j|^2$, then the iteration error in l^2-norm is

$$\|e^{l,k+1}\|^2 = \|S_l^k e^{l,0}\|^2 = \Big\| \sum_{j=1}^{n_l} \rho_j^k a_j \phi_j \Big\|^2 = \sum_{j=1}^{n} \rho_j^{2k} |a_j|^2. \tag{1}$$

Denote the spectrum radius $\rho = \max \rho_j < 1$, then

$$\|e^{l,k+1}\| \le \rho^k \|e^{l,0}\|. \tag{2}$$

In the process of iterations, the higher frequency components with $\rho_j << 1$ is contracted rapidly by several iterations, but the low frequency components with $\rho_j \approx 1$ remain and it is hard to decay. For example, for the efficient CG with spectrum radius $\rho = 1 - ch, c \sim 1$,the error is deduced to only $(1-h)^m \approx 1/e$ by $m = [1/h]$ iterations. This is an essential difficulty.

We see that there are two ways to raise the accuracy, i.e., decrease the spectrum radius or improve the initial value. While CMG is just to provide the good initial value by use of the interpolation and iteration from coarse grids to refine grids.

Assume that the exact solution $\bar{U}^0$ on Z_0 given, CMG has the recurrence algorithms as follows, $l = 1, 2, \cdots, L$.

Step 1) take the linear interpolation $I_1 \bar{U}^{l-1}$ to define the initial value $U^{l,0} = I_1 \bar{U}^{l-1}$;

Step 2) use the operator S_l to get the iterative solution $\bar{U}_l = S_l^{k_l} U^{l,0}$;

Step 3) come back to steps 1) and 2) if $l < L$, until get the final solution $\bar{U}^L$ on Z_L.

The errors in the CMG are measured by the energy norm $\|e\|_K$. Because the gradient of linear finite element is of superconvergence on the (piecewise) uniform grid, the iteration error $\|e\|_K = O(h)$, in general, is easily attained. Thus, CGM is efficient in the energy norm and applicable for many problems, for example, the elastic mechanics. But which is not convergent in L^2-norm and one can get only $\|e\| \le C\|e\|_K = O(h)$ by the embedding theorem, no the optimal error $\|e\| = O(h^2)$. As an example, $e = hx$ in $(0, 1)$, obviously $\|e\| = \|D_x e\| = O(h)$. So CMG is not necessary efficient for the problems measured by $\|e\|$. This surprising phenomenon is observed in our numerical experiments (Of course, if the accuracy $O(h^2)$ in L^2 is not required, we do not need to discuss this question).