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REGULARITY OF ENTROPY SOLUTIONS TO NONCONVEX SCALAR CONSERVATION LAWS WITH MONOTONE INITIAL DATA*

Dedicated to Professor Wu Wenjun on the occasion of his 90th birthday

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Abstract We prove that for a given strictly increasing initial datum in C^k, the solution of the initial value problem is piecewise C^k smooth except for flux functions of nonconvex conservation laws in a certain subset of C^{k+1} of first category, defined in the range of the initial datum.

Key words piecewise smooth solutions; noncovex conservation laws; a set of first category

2000 MR Subject Classification 35L65; 35B65

1 Introduction

Consider the initial value problem for the nonconvex scalar hyperbolic conservation law,

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u = \phi & \text{on } \mathbb{R}\times\{t=0\}. \end{cases} \tag{1.1}$$

We assume that the initial function ϕ is C^k smooth, bounded and satisfies

$$\phi'(x) > 0, \quad \forall x \in \mathbb{R}, \tag{1.2}$$

and the flux function f is C^{k+1} smooth, defined in the range of the initial datum, $3 \le k \le \infty$. In general, the initial value problem (1.1) does not admit a global smooth solution even if the initial datum is smooth, but for arbitrary bounded measurable initial datum a unique global entropy solution does exist.

The piecewise smoothness of entropy solution of convex conservation laws has been studied by many authors, e.g., Chen-Zhang [2], Dafermos [3, 4], Lax [9], Li & Wang [10, 11], Oleinik

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[13], Schaeffer [14], Tadmor & Tassa [15] and Tang & Wang & Zhao [20]. For the structure of nonconvex conservation laws: Dafermos [5] studied the regularity and large time behavior of a conservation law with one inflection point by a direct approach, without making appeal to particular construction scheme; Kruzhkov and Petroyan [8] studied large time behavior of conservation laws using the explicit solution given by Hopf [7]. T.-P. Liu [12] studied admissible solutions of n by n systems of strictly hyperbolic conservation laws and proved that the random choice method approximates discontinuities sharply and yields admissible, in a rough sense, piecewise continuous solutions and the study of the Riemann problem is sufficient in understanding the local and large time behavior of the solution. The results in [12] are new even for scalar conservation law for which one need only to assume the initial data to be of bounded variation and the second derivative of the flux function $f(\cdot)$ has isolated zeros.

The main results of this work will be obtained by the maximization process of $I(x,t,\cdot)$ introduced by Hopf [7] as follows:

$$I(x,t,u)=-\Phi^*(u)+xu-tf(u),\ (x,t)\in\mathbb{R}\times(0,\infty), \tag{1.3}$$

where

$$\Phi^*(u)=\sup_{y\in\mathbb{R}}\,(yu-\Phi(y)),\quad \Phi(y)=\int_0^y\phi(x)\mathrm{d}x.$$

It follows from Bardi and Evans [1] and Kruzhkov and Petroyan [8] that

$$U(x,t)=\sup_{u\in(\phi_-,\phi_+)}I(x,t,u)$$

is the viscosity solution of the initial value problem of Hamilton–Jacobi equation,

$$\begin{cases} U_t+f(U_x)=0 & \text{in } \mathbb{R}\times(0,\infty),\\ U=\Phi & \text{on } \mathbb{R}\times\{t=0\},\end{cases}$$

where $\phi_\pm=\lim\limits_{x\to\pm\infty}\phi(x)$ and then $u(x,t)=U_x(x,t)$, defined almost everywhere in $\mathbb{R}\times(0,\infty)$, is the entropy solution of (1.1).

The mapping $u=\phi(y)$ is one to one and onto from $(-\infty,+\infty)$ to (ϕ_-,ϕ_+) because of (1.2). Then it follows from the fact Hamilton–Jacobi equation is of hyperbolic type with finite domain of dependence that

$$U(x,t)=\sup_{u\in(\phi_-,\ \phi_+)}I(x,t,u)=\sup_{y\in\mathbb{R}}I(x,t,\phi(y))=\max_{y\in\mathbb{R}}I(x,t,\phi(y))=\max_{u\in(\phi_-,\ \phi_+)}I(x,t,u). \tag{1.4}$$

The maximizing value of $I(x,t,\cdot)$, u, must be a critical point of $I(x,t,\cdot)$, the solution of the equation

$$I_u(x,t,u)=-\phi^{-1}(u)+x-tf'(u)=-y+x-tf'(\phi(y))=0. \tag{1.5}$$

In fact in this paper we give an independent proof that $u(x,t)$, the maximizing function of $I(x,t,\cdot)$, is the solution of (1.1). We prove that for a given bounded C^k initial datum satisfying (1.2) then $u(x,t)$, the solution of (1.1), is given by the maximizing function of $I(x,t,\cdot)$ except for flux functions of nonconvex conservation laws in a subset of C^{k+1} of first category, defined in the range of the initial datum and the solution is piecewise smooth.