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ON RITZ METHOD OF AN INTEGRO-DIFFERENTIAL EQUAITON*

Dedicated to Professor Wu Wenjun on the occasion of his 90th birthday

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Abstract This paper deals with the Ritz method of an integro-differential equation related with Riemann zeta-function.

Key words Zeta-function; Ritz method; explicit formula

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Along the line due to Riemann, A. Weil, E. Bombieri [1], an explicit formula for

$$\sum_{\rho} \tilde{f}(\rho)$$

is used to study properties of Riemann zeta-function where $f(x) \in C_0^\infty(0,\infty)$, $\tilde{f}(s)$ denotes the Mellin transformation of $f(x)$, i.e.,

$$\tilde{f}(s) = \int_0^\infty x^{s-1} f(x)\mathrm{d}x,$$

ρ ranges all the complex zeros of Riemann zeta-function.

For our purpose instead of Mellin transformation we use Fourier transformation

$$\hat{F}(t) = \int_{-\infty}^{+\infty} F(\xi)\mathrm{e}^{\mathrm{i}\xi t}\mathrm{d}\xi.$$

We note that if we put $s = \frac{1}{2} + \mathrm{i}t$, $f(\mathrm{e}^\xi)\mathrm{e}^{\frac{\xi}{2}} = F(\xi)$. We have

$$\tilde{f}(s) = \hat{F}(t).$$

So the problem reduces to study the explicit formula for

$$\sum_{\rho} \hat{F}(\gamma), \qquad \rho = \frac{1}{2} + \mathrm{i}\gamma.$$

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1 Explicit Formula

In order to treat variational problem, we consider $F(x) \in \mathring{H}$, where $\mathring{H}$ denotes the Sobolev space which is the closure of $C_0^\infty(-m,m)$ under the norm

$$\|F\|_{\mathring{H}} = \int_{-m}^{m} (|\nabla F|^2 + |F|^2)\mathrm{d}x.$$

In [1], E. Bombieri proved that if we denote $f^*(x) = \frac{1}{x} f(\frac{1}{x})$, then we have

$$\sum_\rho \tilde{f}(\rho) = \tilde{f}(0) + \tilde{f}(1) - \sum \Lambda(n)[f(n) + f^*(n)] - (\log \pi) f(1)$$
$$+\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{\mathrm{i}v}{2}\Big)\tilde{f}\Big(\frac{1}{2} + \mathrm{i}v\Big)\mathrm{d}v.$$

In term of F, this is

$$\sum_\rho \hat{F}(\gamma) = \int_{-\infty}^{+\infty} [F(y) + F(-y)]\mathrm{e}^{\frac{y}{2}}\mathrm{d}y - \sum \Lambda(n)[F(\log n) + F(-\log n)]n^{-\frac{1}{2}}$$
$$-(\log \pi)F(0) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{\mathrm{i}v}{2}\Big)\hat{F}(v)\mathrm{d}v.$$

We use the identity (see [2], Chapter 12)

$$\begin{aligned}\frac{\Gamma'}{\Gamma}(z) &= -\gamma_0 - \frac{1}{z} + \sum_{n=1}^{\infty}\Big(\frac{1}{n} - \frac{1}{n+z}\Big)\\ &= -\gamma_0 + \sum_{n=1}^{N}\frac{1}{n} - \sum_{n=0}^{N}\frac{1}{n+z} + \sum_{n=N+1}^{\infty}\Big(\frac{1}{n} - \frac{1}{n+z}\Big)\\ &= p_N(z) + r_N(z),\end{aligned}$$

where $\gamma_0 =$ Euler constant. From [1],

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{Re} p_N\Big(\frac{1}{4} + \frac{\mathrm{i}v}{2}\Big)\hat{F}(v)\mathrm{d}v\\ =& \frac{1}{2\pi}\int_{-\infty}^{+\infty}\Big(-\gamma_0 + \sum_{n=1}^{N}\frac{1}{n} - \sum_{n=0}^{N}\frac{4n+1}{(2n+\frac{1}{2})^2 + v^2}\Big)\hat{F}(v)\mathrm{d}v\\ =& (-\gamma_0 + \zeta_N)F(0) - \sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{4n+1}{(2n+\frac{1}{2})^2+v^2}\hat{F}(v)\mathrm{d}v.\end{aligned}$$

By

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{2a}{a^2+v^2}\hat{F}(v)\mathrm{d}v = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}F(\xi)\mathrm{d}\xi\int_{-\infty}^{+\infty}\Big(\frac{1}{v-\mathrm{i}a} - \frac{1}{v+\mathrm{i}a}\Big)\mathrm{e}^{\mathrm{i}v\xi}\mathrm{d}v,$$
$$\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\Big(\frac{1}{v-\mathrm{i}a} - \frac{1}{v+\mathrm{i}a}\Big)\mathrm{e}^{\mathrm{i}v\xi}\mathrm{d}v = \min\{\mathrm{e}^{\xi}, \mathrm{e}^{-\xi}\}^a.$$

So

$$\begin{aligned}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{4n+1}{(2n+\frac{1}{2})^2+v^2}\hat{F}(v)\mathrm{d}v &= \int_0^{+\infty}F(\xi)\mathrm{e}^{-(2n+\frac{1}{2})\xi}\mathrm{d}\xi + \int_{-\infty}^{0}F(\xi)\mathrm{e}^{(2n+\frac{1}{2})\xi}\mathrm{d}\xi\\ &= \sum_{n=0}^{N}\int_0^{+\infty}[F(\xi) + F(-\xi)]\mathrm{e}^{-(2n+\frac{1}{2})\xi}\mathrm{d}\xi.\end{aligned}$$