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# BIHARMONIC EQUATIONS WITH ASYMPTOTICALLY LINEAR NONLINEARITIES*

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**Abstract** This article considers the equation

$$\Delta^2 u = f(x,u)$$

with boundary conditions either $u|_{\partial\Omega} = \frac{\partial u}{\partial n}|_{\partial\Omega} = 0$ or $u|_{\partial\Omega} = \triangle u|_{\partial\Omega} = 0$, where $f(x,t)$ is asymptotically linear with respect to $t$ at infinity, and $\Omega$ is a smooth bounded domain in $R^N, N > 4$. By a variant version of Mountain Pass Theorem, it is proved that the above problems have a nontrivial solution under suitable assumptions of $f(x,t)$.

**Key words** Biharmonic, mountain pass theorem, asymptotically linear

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## 1 Introduction and Main Results

Considering the following biharmonic problems:

$$\text{(P1)} \qquad \begin{cases} \Delta^2 u = f(x,u), & \text{in } \Omega; \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0, \end{cases}$$

and

$$\text{(P2)} \qquad \begin{cases} \Delta^2 u = f(x,u), & \text{in } \Omega; \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \end{cases}$$

where $\Omega$ is a smooth bounded domain in $R^N, N > 4$. We assume that $f(x,t)$ satisfies the following hypotheses:

(H1) $f(x,t) \in C(\bar{\Omega} \times R); f(x,0) = 0, \forall x \in \bar{\Omega}; f(x,t) \geq 0, \forall t \geq 0, x \in \bar{\Omega}$ and $f(x,t) \equiv 0, \forall t \leq 0, x \in \bar{\Omega}$.

(H2) $\lim\limits_{t\to 0} \frac{f(x,t)}{t} = p(x), \lim\limits_{t\to +\infty} \frac{f(x,t)}{t} = l$ $(0 < l \leq +\infty)$ uniformly in a.e.$x \in \Omega$, where $0 \leq p(x) \in L^\infty(\Omega)$, and $|p(x)|_\infty < \lambda_1$ for the problem $(P1), \lambda_1$ is the first eigenvalue of

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$(\Delta^2, H_0^2(\Omega)); |p(x)|_\infty < \Lambda_1$ for the problem (P2), $\Lambda_1$ is the first eigenvalue of $(\Delta^2, H^2(\Omega) \cap H_0^1(\Omega))$.

By (H2), we call $f(x,t)$ asymptotically linear with respect to $t$ at infinity.

**Definition** (1) $u \in H_0^2(\Omega)$ is a weak solution to (P1) if

$$\int_\Omega \Delta u \Delta \varphi \mathrm{d}x = \int_\Omega f(x,u)\varphi \mathrm{d}x \text{ for all } \varphi \in H_0^2(\Omega); \tag{1.1}$$

(2) $u \in H^2(\Omega) \cap H_0^1(\Omega)$ is a weak solution to (P2) if

$$\int_\Omega \Delta u \Delta \varphi \mathrm{d}x = \int_\Omega f(x,u)\varphi \mathrm{d}x \text{ for all } \varphi \in H^2(\Omega) \cap H_0^1(\Omega); \tag{1.2}$$

where $H_0^2(\Omega)$ is the standard Sobolev space with equivalent norm $\|\Delta u\|_{L^2(\Omega)}$ according to Gilbarg and Trudinger [1, Corollary 9.10], and the Hilbert space $H^2(\Omega) \cap H_0^1(\Omega)$ can be endowed with the scalar product $(u,v) = \int_\Omega \Delta u \Delta v \mathrm{d}x$, which induces the norm $\|u\| = \|\Delta u\|_{L^2(\Omega)}$ equivalent to the standard norm of $H^2(\Omega)$(see [2], Remark 2.1 and 2.2). For the sake of completeness we will give a short proof in Section 2.

**Remark 1.1** For Dirichlet boundary value problem (P1), it is well-known that seeking a weak solution of (P1) is equivalent to finding a nonzero critical point of the following functional on $H_0^2(\Omega)$:

$$I(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 \mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x, \text{ where } F(x,u) = \int_0^u f(x,t)\mathrm{d}t. \tag{1.3}$$

But for the Navier boundary value problem (P2), it is not so clear why the second boundary condition can be satisfied by a critical point of the following functional on $H^2(\Omega) \cap H_0^1(\Omega)$ :

$$J(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 \mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x, \text{ where } F(x,u) = \int_0^u f(x,t)\mathrm{d}t. \tag{1.4}$$

In [3] using some information about the regularity of such critical points the author showed that the critical point for $J$ satisfies the second boundary condition in the sense of traces.

In order to find critical points of the functional $I$ in (1.3) (or $J$ in (1.4)), one usually applies the Mountain Pass Theorem proposed by Ambrosetti and Rabinowitz in [11]. For applying the theorem, one often requires the following technique condition introduced in [11], that is, for some $\theta > 2$ and $M > 0$,

$$0 < \theta F(x,t) \le f(x,t)t \text{ for all } |t| \ge M \text{ and } x \in \Omega. \tag{AR}$$

Condition (AR) is important for ensuring that the functional $I$ in (1.3) (or $J$ in (1.4)) has a "Mountain Pass" geometry and that each $(\mathrm{PS})_\mathrm{c}$ sequence is bounded in $H_0^2(\Omega)$ (or $H^2(\Omega) \cap H_0^1(\Omega)$).

If $f(x,t)$ admits subcritical growth and satisfies (AR) condition by the standard argument of applying Mountain Pass Theorem, we know that problems (P1) and (P2) have nontrivial solutions. Similarly, lase $f(x,t)$ is of critical growth (See, for example, [3–6], [13], and their references). Recently, problem (P1) was also studied by Yao and Shen [14] for $f(x,u)$ to be just linear.