Available online at www.sciencedirect.com

SCIENCE DIRECT®

Acta Mathematica Scientia 2006,**26B**(4):720–728

Acta Mathematica Scientia 数学物理学报

www.wipm.ac.cn/publish/

RELATIVE WIDTH OF SMOOTH CLASSES OF MULTIVARIATE PERIODIC FUNCTIONS WITH RESTRICTIONS ON ITERATED LAPLACE DERIVATIVES IN THE L_2-METRIC*

Liu Yongping (刘永平)
School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China
E-mail: ypliu@bnu.edu.cn

Yang Lianhong (杨连红)
Department of Applied Mathematics, Communication University of China, Beijing 100024, China
E-mail: yhongl@163.com

Abstract For two subsets W and V of a Banach space X, let $K_n(W,V,X)$ denote the relative Kolmogorov n-width of W relative to V defined by

$$K_n(W,V,X) := \inf_{L_n} \sup_{f\in W} \inf_{g\in V\cap L_n} \| f-g \|_X,$$

where the infimum is taken over all n-dimensional linear subspaces L_n of X. Let $W_2(\Delta^r)$ denote the class of 2π-periodic functions f with d-variables satisfying

$$\int_{[-\pi,\pi]^d} |\Delta^r f(x)|^2 \,\mathrm{d}x \le 1,$$

while Δ^r is the r-iterate of Laplace operator Δ. This article discusses the relative Kolmogorov n-width of $W_2(\Delta^r)$ relative to $W_2(\Delta^r)$ in $L_q([-\pi,\pi]^d)$ $(1\le q\le\infty)$, and obtain its weak asymptotic result.

Key words Multivariate function classes, width, relative width

2000 MR Subject Classification 41A46, 41A65, 46E35

1 Introduction

The notion of relative width has been introduced by V. N. Konovalov [1]. Let W and V be centrally symmetric sets in a Banach space X. The relative Kolmogorov n-width of W relative to V is defined by

$$K_n(W,V,X) := \inf_{L_n} \sup_{f\in W} \inf_{g\in V\cap L_n} \| f-g \|_X,$$

*Received November 30, 2004. Supported partly by National Natural Science Foundation of China (10471010) and partly by the project "Representation Theory and Related Topics" of the "985 Program" of Beijing Normal University

where the infimum is taken over all n-dimensional subspaces L_n of X.

When $V = X$, the relative Kolmogorov n-width $K_n(W, X, X)$ coincides with the usual Kolmogorov n-width of W in X, which we shall denote by $d_n(W, X)$. Obviously, $K_n(W, V, X) \geq d_n(W, X)$ for any set $V \subset X$.

Let $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{Z}_+$ denote the set of all real numbers, all integral numbers, and all positive integral numbers, respectively. For $1 \leq p \leq \infty$ and $d \in \mathbb{Z}_+$, let $L_p(T^d)$ ($T^d = [-\pi, \pi]^d$, and $T = T^1$) denote the real space of 2π periodic Lebesgue measurable functions f with the following finite norm

$$\|f\|_{L_p(T^d)} = \left\{ \int_{T^d} |f(x)|^p \, \mathrm{d}x \right\}^{1/p}, \ 1 \leq p < \infty;$$

$$\|f\|_{L_\infty(T^d)} = \operatorname*{ess\ sup}_{x \in T^d} |f(x)|, \ p = \infty.$$

When $d = 1$ and $r \in \mathbb{Z}_+$, let $W_p^r(T)$ denote the Sobolev class of 2π periodic continuous functions f for which the $(r-1)$-th derivatives are absolutely continuous and the r-th derivatives satisfy $\|f^{(r)}\|_{L_p(T)} \leq 1$.

Konovalov in [1] showed that for all $r \in \mathbb{Z}_+$ and $r \geq 2$ there holds

$$K_n(W_\infty^r(T), W_\infty^r(T), L_\infty(T)) \asymp n^{-2}, \ n \to \infty,$$

$$K_n(W_\infty^1(T), W_\infty^1(T), L_\infty(T)) \asymp n^{-1}, \ n \to \infty.$$

Babenko in [2] established a similar result for the class $W_1^r(T)$ in the $L_1(T)$-metric as follows $K_n(W_1^r(T), W_1^r(T), L_1(T)) \asymp n^{-2}, n \to \infty$, for $r \in \mathbb{Z}_+$ and $r \geq 3$.

Konovalov in [3] obtained that for each $r \in \mathbb{Z}_+$ and $1 \leq q \leq \infty$,

$$K_n(W_2^r(T), W_2^r(T), L_q(T)) \asymp n^{-\min\{r - \frac{1}{2} + \frac{1}{q}, r\}}, \quad n \geq 1.$$

It is well-known (see, for example, [4, p.249]) that for the univariate functions, for all $r \in \mathbb{Z}_+$,

$$K_n(W_2^r(T), W_2^r(T), L_2(T)) \asymp n^{-r}, \quad n \to \infty.$$

When $d > 1$, for a given vector $\mathbf{r} = (r_1, r_2, \cdots, r_d)$ with natural components and such that $r_j \geq 3$ $(1 \leq j \leq d)$, set $W_p^{\mathbf{r}} = \{f \in L_p(T^d) : \|D^{\mathbf{r}} f\|_{L_p(T^d)} \leq 1\}$ and

$$W_*^{\mathbf{r}} L_p(T^d) = \{B_{\mathbf{r}} * \phi : \ \|\phi\|_{L_p(T^d)} \leq 1, \int_T \phi(x) \, \mathrm{d}x_j = 0, j = 1, \cdots, d\},$$

where $D^{\mathbf{r}} = \frac{\partial^{r_1 + r_2 + \cdots + r_d}}{\partial^{r_1} x_1 \partial^{r_2} x_2 \cdots \partial^{r_d} x_d}$, $B_{\mathbf{r}}(x) = \prod_{j=1}^{d} B_{r_j}(x_j), x = (x_1, x_2, \cdots, x_d)$, while

$$B_m(t) = \pi^{-1} \sum_{k=1}^{\infty} k^{-m} \cos(kt - \frac{m\pi}{2}), \ t \in \mathbb{R}, m \in \mathbb{Z}_+,$$

and $B_{\mathbf{r}} * \phi$ denotes the convolution of $B_{\mathbf{r}}$ and ϕ defined by

$$B_{\mathbf{r}} * \phi(x) = \int_{T^d} B_{\mathbf{r}}(x - y) \phi(y) \mathrm{d}y.$$

Babenko in [5] generalized the result of [2] to the multivariate case and obtained the following estimate $K_n(W_*^{\mathbf{r}} L_1(T^d), W_1^{\mathbf{r}}(T^d), L_1(T^d)) \asymp n^{-2/d}, n \to \infty$.