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BRUNN-MINKOWSKI INEQUALITY FOR VARIATIONAL FUNCTIONAL INVOLVING THE *P*-LAPLACIAN OPERATOR*

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Abstract In this paper, we investigate the following elliptic problem involving the P-Laplacian

$$(\mathrm{P})\quad \begin{cases} -\mathrm{Div}(|\nabla u|^{p-2}\nabla u)=|u|^{q-1}u & \text{in } K,\\ u>0 & \text{in } K,\\ u=0 & \text{on } \partial K,\end{cases}$$

where $p>1, 0<q<p-1, K\subset R^n$ with $\overline{K}\in\mathcal{K}^n$, and prove that the energy integral of the problem (P) satisfies a Brunn-Minkowski type inequality.

Key words P-Laplacian; energy integral; Brunn-Minkowski type inequlity

2000 MR Subject Classification 35J20; 35J60

1 Introduction and Main Result

The intention of this paper is to investigate Brunn-Minkowski type inequality for energy integral of solution to the following elliptic problem involving the P-Laplacian

$$\begin{cases} -\mathrm{Div}(|\nabla u|^{p-2}\nabla u)=|u|^{q-1}u & \text{in } K,\\ u>0 & \text{in } K,\\ u=0 & \text{on } \partial K,\end{cases}\tag{1}$$

where $p>1, 0<q<p-1$ and K is an open set in R^n with $\overline{K}\in\mathcal{K}^n$, here $\mathcal{K}^n$ denotes the set of convex bodies (compact convex set with nonempty interiors) in R^n. The variational functional corresponding to (1) is

$$F(u)=\frac{1}{p}\int_K|\nabla u|^p\mathrm{d}x-\frac{1}{q+1}\int_K|u|^{q+1}\mathrm{d}x,\tag{2}$$

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where $u \in W_0^{1,p}(K)$. If u is a solution to (1) in K, we define the energy integral of u with respect to K by

$$E(K) = \int_K |\nabla u|^p \mathrm{d}x. \tag{3}$$

In order to state our main result, we first give some notations.

Throughout this paper, we use following notations: K and K_i $(i = 0, 1, t)$ denote open sets in R^n with the closures $\overline{K}, \overline{K}_i \in \mathcal{K}^n$, respectively. We also fix $m = p - q - 1$, where p and q are given in (1).

Our goal is to prove that $E(K)$ satisfies a Brunn-Minkowski inequality.

The classical Brunn-Minkowski inequality states that, if $K_0, K_1 \in R^n$ are convex sets with compact closure, and $t \in [0, 1]$, then

$$V^{\frac{1}{n}}((1-t)K_0 + tK_1) \geq (1-t)V^{\frac{1}{n}}(K_0) + tV^{\frac{1}{n}}(K_1),$$

where $(1-t)K_0 + tK_1 = \{(1-t)x + ty : x \in K_0, y \in K_1\}$ and $V(K)$ is the n-dimensional Lebesgue measure of K. The equality holds if and only if K_0 is homothetic to K_1, that is, $K_1 = x + tK_0$ for some $x \in R^n, t > 0$.

For more details about the Brunn-Minkowski inequality, the readers can consult the book [25] written by Schneider and the survey article [16] by Gardner. The Brunn-Minkowski inequality is one of the most and deepest results in the theory of convex bodies [25], and it is associated with other fundamental inequalities such as the isoperimetric inequality, the Sobolev inequality and the Prékopa-Leindler inequality [16]. It can also be applied to other problems of convexity including convex functions, convex geometry, probability theory on convex sets, and computational complexity, see for example, in [1], [3], [4], [16], [20], [21], [25], [26], [29], [31] and references therein.

In [11], Colesanti collected three main examples for Brunn-Minikowski inequalities: the Brunn-Minkowski inequality for the first eigenvalue of the Laplace operator (see details in [5], [9]), the Newtonian capacity (in [6], [7], [10], [12]) and the torsional rigidity (in [8]).

The Brunn-Minkowski inequalities were extended in various directions in [12], [14], [24]. Very recently, Colesanti studied in [11] the following problem:

$$\begin{cases} \Delta u = -u^p & \text{in } K, \\ u > 0 & \text{in } K, \\ u = 0 & \text{on } \partial K, \end{cases} \tag{4}$$

where $K \in R^n, 0 < p < 1$, and proved that the energy integral

$$F(K) = \int_K |\nabla u|^2 \mathrm{d}x$$

of the solution to (4) satisfies a Brunn-Minkowski inequality. Later, Colesanti, Cuoghi and Salani [13] proved the Poincaré constant $\lambda(K)$ of domain $K \in R^n$, defined by

$$\lambda(K) = \inf\left\{\frac{\int_K |\nabla v|^p \mathrm{d}x}{\int_K |v|^p \mathrm{d}x} : v \in W_0^{1,p}(K), \int_K |v|^p \mathrm{d}x > 0\right\}, \tag{5}$$

verifies a Brunn-Minkowski inequality: