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MULTIPLE SOLUTIONS FOR SCHRÖDINGER EQUATIONS WITH MAGNETIC FIELD*

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Abstract The authors consider the semilinear Schrödinger equation

$$-\Delta_A u + V_\lambda(x)u = Q(x)|u|^{\gamma-2}u \quad \text{in} \quad \mathbb{R}^N,$$

where $1 < \gamma < 2^*$ and $\gamma \neq 2$, $V_\lambda = V^+ - \lambda V^-$. Exploiting the relation between the Nehari manifold and fibrering maps, the existence of nontrivial solutions for the problem is discussed.

Key words Nehari manifold; fibrering maps; Schrödinger equation

2000 MR Subject Classification 35J60; 35J65

1 Introduction

In this article, we study the existence of nontrivial solutions of the semilinear Schrödinger equation

$$-\Delta_A u + V_\lambda(x)u = Q(x)|u|^{\gamma-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $-\Delta_A = (-i\nabla + A)^2$, $u : \mathbb{R}^N \to \mathbb{C}$, $N \geq 3$, $1 < \gamma < 2^*$ and $\gamma \neq 2$. The coefficient V_λ is the scalar (or electric) potential and $A = (A_1, \cdots, A_N) : \mathbb{R}^N \to \mathbb{R}^N$ is the vector (or magnetic) potential. We assume in this paper that $A \in L^2_{\rm loc}(\mathbb{R}^N)$, $V_\lambda(x)$ and $Q(x)$ are continuous functions changing signs on $\mathbb{R}^N$. $V_\lambda(x) = V^+(x) - \lambda V^-(x)$, where $V^+(x) = \max(V(x), 0)$, $V^-(x) = \max(-V(x), 0)$ and $V^-(x) \in L^{\frac{N}{2}}(\mathbb{R}^N)$. It is assumed that $\lim\limits_{|x|\to\infty} Q(x) = Q(\infty) < 0$. Further assumptions on $V_\lambda(x)$ and $Q(x)$ will be formulated later.

In the case $A = 0$, the problem was extensively studied. In particular, in a bounded domain Ω, it was established in [4] the existence and multiplicity of non-negative solutions of (1.1) for $\gamma > 2$. Later, the case $1 < \gamma < 2$ was considered in [2]. In the whole space $\mathbb{R}^N$, if $V \in L^{\frac{N}{2}}(\mathbb{R}^N)$, the eigenvalue problem

$$-\Delta u = \lambda V(x)u \quad \text{in } \mathbb{R}^N$$

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has a sequence of eigenvalues, $0 < \lambda_1(V) \le \lambda_2(V) \le \cdots \le \lambda_n(V) \le \cdots$, of finite multiplicity and going to infinity. Under this condition, it was proved in [8], that, for $V_\lambda = -\lambda V$,

(i) problem (1.1) has a positive solution for every $0 < \lambda < \lambda_1(V)$.

(ii) if $\int_{\mathbb{R}^N} Q\phi_1^\gamma \mathrm{d}x < 0$, where ϕ_1 is the eigenfunction corresponding to $\lambda_1(V)$, then there exists a constant $\delta > 0$ such that problem (1.1) admits at least two positive solutions for every $\lambda_1(V) < \lambda < \lambda_1(V) + \delta$. These solutions were obtained by the mountain-pass lemma and local minimization. Similar results were obtained for the $p-$Laplacian in $\mathbb{R}^N$ in [6] and [10].

Recently, much interest in the case $A \neq 0$ has arisen and various existence results were obtained, see for instance, [1], [5], [7], [11] and references therein. Inspired by [2] and [4], we consider the existence of nontrivial solutions for (1.1) with $A \neq 0$. We classify the Nehari manifold, and find solutions of (1.1) as minimizers of the associated functional on two distinct components of the Nehari manifold.

It is known from [7] that the eigenvalue problem

$$-\Delta_A u + V^+(x)u = \mu V^-(x)u \qquad \text{in } \mathbb{R}^N \tag{1.2}$$

has a sequence of eigenvalues $0 < \mu_1 < \mu_2 \le \mu_3 \le \ldots \le \mu_n \to \infty$ if $V^- \neq 0$ and $V^- \in L^{\frac{N}{2}}(\mathbb{R}^N)$. Let us denote the corresponding orthonormal system of eigenfunctions by $\varphi_1(x), \varphi_2(x), \cdots$. The sequence is complete in the Hilbert space $H^1_{A,V^+}(\mathbb{R}^N)$, where $H^1_{A,V^+}(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} \left(|\nabla_A u|^2 + V^+(x)|u|^2 \right) \mathrm{d}x \Big)^{\frac{1}{2}},$$

and $\nabla_A u = (\nabla + iA)u$, $V^+(x) = \max(V(x), 0)$. The first eigenvalue μ_1 is defined by the Rayleigh quotient

$$\mu_1 = \inf_{u \in H^1_{A,V^+}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} (|\nabla_A u|^2 + V^+(x)|u|^2)\mathrm{d}x}{\int_{\mathbb{R}^N} V^-(x)|u|^2 \mathrm{d}x}. \tag{1.3}$$

Our main result is as follows.

Theorem 1.1 If $2 < \gamma < 2^*$, then

(i) problem (1.1) has a solution for $0 < \lambda < \mu_1$.

(ii) if $\int_{\mathbb{R}^N} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$ and $\lambda = \mu_1$, then problem (1.1) has a solution.

(iii) if $\int_{\mathbb{R}^N} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$, then there exists a constant $\delta > 0$ such that problem (1.1) admits at least two solutions for $\mu_1 < \lambda < \mu_1 + \delta$.

For the case of $1 < \gamma < 2$, we have

Theorem 1.2 (i) Problem (1.1) has a solution for $0 < \lambda < \mu_1$.

(ii) If $\int_{\mathbb{R}^N} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$, then there exists a constant $\delta > 0$ such that problem (1.1) admits at least two solutions for $\mu_1 < \lambda < \mu_1 + \delta$.

We point out that φ_1 may not belong to $L^\gamma(\mathbb{R}^N)$. The condition $\int_{\mathbb{R}^N} Q(x)|\varphi_1|^\gamma \mathrm{d}x < 0$ is an extra assumption on Q .

In Section 2 we discuss the relation between the Nehari manifold and fibrering maps. Theorems 1.1 and 1.2 are proved in Sections 3 and 4, respectively.