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REGULARITY OF THE SOLUTIONS FOR NONLINEAR BIHARMONIC EQUATIONS IN $\mathbb{R}^N$*

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Abstract The purpose of this article is to establish the regularity of the weak solutions for the nonlinear biharmonic equation

$$\begin{cases} \Delta^2 u + a(x)u = g(x,u), \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

where the condition $u \in H^2(\mathbb{R}^N)$ plays the role of a boundary value condition, and as well expresses explicitly that the differential equation is to be satisfied in the weak sense.

Key words nonlinear biharmonic equation; regularity; fundamental solutions

2000 MR Subject Classification 35B40; 35B65; 35G20; 35J60; 35J30

1 Introduction

The purpose of this article is to establish the regularity of the weak solutions for a certain nonlinear biharmonic equation in $\mathbb{R}^N$ $(N \geq 1)$. We consider solutions $u: \mathbb{R}^N \to \mathbb{R}$ of the problem

$$\begin{cases} \Delta^2 u + a(x)u = g(x,u), \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where the condition $u \in H^2(\mathbb{R}^N)$ plays the role of a boundary value condition, and as well expresses explicitly that the differential equation is to be satisfied in the weak sense. We assume that

H_1) $g(x,u): \mathbb{R}^N \times \mathbb{R}^1 \to \mathbb{R}^1$ is measurable in x and continuous in u, and $\sup\limits_{\substack{x\in\mathbb{R}^N \\ 0\leq u\leq M}} |g(x,u)| < \infty$ for every $M > 0$;

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H$_2$) there exist two constants $\sigma, \delta, \sigma > \delta > 0$, and two functions $b_1(x), b_2(x) \in L^\infty(\mathbb{R}^N)$ such that $|g(x,u)| \le b_1(x)|u|^{\delta+1} + b_2(x)|u|^{\sigma+1}$;

H$_3$) $\lim\limits_{|x|\to\infty} a(x) = k^2$ with $k > 0$ and $(k^2 - a(x)) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Then we have the following theorem:

Theorem 1.1 Assume that H_1) to H_3) hold with $\sigma+1 < \frac{N+4}{N-4}$ if $N \ge 5$ and $\sigma+1 < +\infty$ if $1 \le N \le 4$. Let u be a weak solution of (1.1). Then $u \in H^4(\mathbb{R}^N) \cap W^{2,s}(\mathbb{R}^N)$ for $2 \le s \le +\infty$. In particular, $u \in C^2(\mathbb{R}^N)$ and $\lim\limits_{|x|\to\infty} u(x) = 0$, $\lim\limits_{|x|\to\infty} \Delta u(x) = 0$.

Dealing with regularity of solutions is much more complicated for biharmonic equations than for problems that can be treated by well-developed standard methods, such as second-order elliptic problems. First of all, there is no maximum principle for the biharmonic operator. So we can't get some estimates of the solutions by the methods used in dealing with second-order elliptic problems. Secondly, we know little about the properties of the eigenfunctions of the biharmonic operator in $\mathbb{R}^N$. To overcome these difficulties, we first introduce the fundamental solutions for the linear biharmonic operator $\Delta^2 + k^2$ for $k > 0$. By applying some properties of Hankel functions, which are the solutions of Bessel' equation, we obtain the asymptotic representation of the fundamental solution of $\Delta^2 + k^2$ at ∞ and 0. Then we prove that, for $p > 1$,

$$\Delta^2 - \lambda: \quad W^{2,p}(\mathbb{R}^N) \longrightarrow L^p(\mathbb{R}^N)$$

is an isomorphism if $\lambda < 0$. Some estimates of the solutions of (1.1) can be obtained from the properties of the fundamental solutions of operator $\Delta^2 - \lambda$. We also establish some L^p theory for the biharmonic problem (1.1) so that a bootstrap argument can be used to deduce the regularity of the solutions of the biharmonic problem (1.1). Please refer to Grunau [3], Jannelli [5], Noussair, Swanson and Yang [8], Peletier and Van der Vorst [9], Pucci and Serrin [10], Yao [14] for the early results on the existence and properties of solutions associated with biharmonic operators.

The organization of this article is as follows: In Section 2, we introduce the fundamental solutions of $\Delta^2 - \lambda$ for $\lambda < 0$ and establish some properties of these fundamental solutions. In Section 3, we show that a weak solution of the linear problem

$$\begin{cases} \Delta^2 u - \lambda u = f(x), \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{1.2}$$

belongs to $H^4(\mathbb{R}^N)$ whenever $f \in L^2(\mathbb{R}^N)$. In Section 4, we obtain a sharper relationship between the regularity of the weak solutions of the linear biharmonic problem (1.2) and the properties of the inhomogeneous term f in (1.2). In Section 5, we establish the regularity of the weak solutions for the nonlinear problem (1.1).

2 Fundamental Solutions

In this section, we give some properties of the fundamental solutions for the biharmonic operator $\Delta^2 + k^2$. The proof of these properties can be found in [2].

Lemma 2.1 Let $G_k^{(N)}(|x|)$ be the fundamental solutions of biharmonic operator $\triangle^2 + k^2$ for $k > 0$ and $g_\delta^{(N)}(|x|)$ be the fundamental solutions of Laplace operator $-\triangle + \delta$. Then we