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Cone-volume measure and stability ☆

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ABSTRACT

We prove stability results for two central inequalities involving the cone-volume measure of a centered convex body: the subspace concentration conditions and the U-functional/volume inequality.

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1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies in $\mathbb{R}^n$ having non-empty interiors, i.e., $K \in \mathcal{K}^n$ is a convex compact subset of the n-dimensional Euclidean space $\mathbb{R}^n$ with $\operatorname{int}(K) \neq \emptyset$. As usual, we denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{R}^n \times \mathbb{R}^n$ with associated Euclidean norm $\| \cdot \|$, and $S^{n-1} \subset \mathbb{R}^n$ denotes the $(n-1)$-dimensional unit sphere, i.e., $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

For $K \in \mathcal{K}^n$ we write $S_K(\cdot)$ and $h_K(\cdot)$ to denote its surface area measure and support function, respectively, and ν_K to denote the Gauß map assigning the outer unit normal $\nu_K(x)$ to an $x \in \partial_* K$, where $\partial_* K$ consists of all points in the boundary ∂K of K having a unique outer normal vector (cf. Section 2 for precise definitions). If the origin o lies in $K \in \mathcal{K}^n$, the *cone-volume measure of* K on S^{n-1} is given by

$$\mathrm{V}_K(\omega) = \int_{\omega} \frac{h_K(u)}{n} \, \mathrm{d}S_K(u) = \int_{\nu_K^{-1}(\omega)} \frac{\langle x, \nu_K(x) \rangle}{n} \, \mathrm{d}\mathcal{H}_{n-1}(x), \tag{1.1}$$

where $\omega \subseteq S^{n-1}$ is a Borel set and, in general, $\mathcal{H}_k(x)$ denotes the k-dimensional Hausdorff measure. Instead of $\mathcal{H}_n(\cdot)$, we also write $\mathrm{V}(\cdot)$ for the n-dimensional volume.

The name cone-volume measure stems from the fact that if K is a polytope with facets $F_1, \ldots, F_m$ and corresponding outer unit normals $u_1, \ldots, u_m$, then

$$\mathrm{V}_K(\omega) = \sum_{i=1}^{m} \mathrm{V}([o, F_i]) \delta_{u_i}(\omega). \tag{1.2}$$

Here δ_{u_i} is the Dirac delta measure on S^{n-1} concentrated at u_i, and for $x_1, \ldots, x_m \in \mathbb{R}^n$ and subsets $S_1, \ldots, S_l \subseteq \mathbb{R}^n$ we denote the convex hull of the set $\{x_1, \ldots, x_m, S_1, \ldots, S_l\}$ by $[x_1, \ldots, x_m, S_1, \ldots, S_l]$. With this notation $[o, F_i]$ is the cone with apex o and basis F_i.

In recent years, cone-volume measures have appeared and were studied in various contexts, see, e.g., F. Barthe, O. Guedon, S. Mendelson and A. Naor [3], K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang [8,9], R. Gardner, D. Hug and W. Weil [20], M. Gromov and V.D. Milman [22], M. Ludwig [35], M. Ludwig and M. Reitzner [36], E. Lutwak, D. Yang and G. Zhang [41], A. Naor [44], A. Naor and D. Romik [45], G. Paouris and E. Werner [46], A. Stancu [51], G. Zhu [54,55], K.J. Böröczky and P. Hegedűs [5], L. Ma [43], Y. Huang, E. Lutwak, D. Yang and G. Zhang [32].

In particular, cone-volume measures are the subject of the *logarithmic Minkowski problem*. This is the particular interesting case $p = 0$ of the general L_p-Minkowski problem which is at the core of the L_p-Brunn–Minkowski theory, one of the cornerstones of modern convex geometry. The L_p-Minkowski problem asks for a characterization of the L_p surface area measure