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Symmetrization in geometry ☆

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ABSTRACT

The concept of an i-symmetrization is introduced, which provides a convenient framework for most of the familiar symmetrization processes on convex sets. Various properties of i-symmetrizations are introduced and the relations between them investigated. New expressions are provided for the Steiner and Minkowski symmetrals of a compact convex set which exhibit a dual relationship between them. Characterizations of Steiner, Minkowski and central symmetrization, in terms of natural properties that they enjoy, are given and examples are provided to show that none of the assumptions made can be dropped or significantly weakened. Other familiar symmetrizations, such as Schwarz symmetrization, are discussed and several new ones introduced.

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1. Introduction

Around 1836, Jakob Steiner introduced the process now known as Steiner symmetrization in attempting to prove the isoperimetric inequality. His proof was incomplete, since he assumed the existence of the extremum, but a standard modern approach (see, for example, [12, Chapter 9]) is still based on Steiner symmetrization. Indeed, Steiner symmetrization remains an extremely potent technique in geometry, where it has found frequent use, for instance in the demonstration of a variety of powerful affine isoperimetric inequalities. See, for example, [8, Chapter 9], [12, Chapter 9], [29, Chapter 10], and the references given there. Beyond geometry, Steiner symmetrization plays an important role in several areas of mathematics, particularly analysis and PDEs. The latter development was stimulated by the appearance of the classic text of Pólya and Szegö [26], which inspired a huge number of works. See, for example, [2,6,14–17,19,20,30], and the references given in these texts.

Despite the vast literature surrounding Steiner symmetrization and its applications, we are not aware of a characterization of it, and one purpose of this paper is to provide some. We also formulate a general framework for many symmetrizations: For $i \in \{0, \ldots, n-1\}$ and an i-dimensional subspace H in $\mathbb{R}^n$, we call a map $\diamondsuit$, from a class $\mathcal{B}$ of nonempty compact sets in $\mathbb{R}^n$ to the subclass $\mathcal{B}_H$ of members of $\mathcal{B}$ that are H-symmetric (i.e., symmetric with respect to H), an *i-symmetrization* on $\mathcal{B}$. With this terminology in place, we show that *Steiner symmetrization is the unique $(n-1)$-symmetrization on convex bodies in $\mathbb{R}^n$, $n \geq 2$, that is monotonic, volume preserving, and either invariant on H-symmetric spherical cylinders or projection invariant.* (See Section 3 for the definitions of these properties and Section 2 for basic terminology and notation.) The version assuming invariance on H-symmetric spherical cylinders is a consequence of a result we prove for Steiner symmetrization on compact sets in $\mathbb{R}^n$, $n \geq 2$. Examples are given that suggest that the familiar generalization of Steiner symmetrization called Schwarz symmetrization may be difficult to classify in a nontrivial manner.

Another process familiar in geometry is now usually called Minkowski symmetrization, despite being introduced by Blaschke (see [12, p. 174] and [29, p. 181]), because up to a scaling factor it involves taking the Minkowski sum of a set and its reflection in a subspace. The significance of the Minkowski symmetral of a compact convex set stems partly from the fact that it contains the Steiner symmetral of the set. This relationship has been found particularly useful in studying the convergence of successive Steiner symmetrals. See, for example, [5], [12, Chapter 9], [29, Notes for Sections 3.3 and 10.3], and the references there for the many deep results on this topic by various authors. We prove that *Minkowski symmetrization is the unique $(n-1)$-symmetrization on convex bodies (or on compact convex sets) in $\mathbb{R}^n$, $n \geq 2$, that is monotonic, mean width preserving, and either invariant on H-symmetric spherical cylinders or projection invariant.*

The paper is structured as follows. In Section 3, as well as introducing i-symmetrizations, we define the eight main properties of them that we find the most useful and