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Schubert polynomials, slide polynomials, Stanley symmetric functions and quasi-Yamanouchi pipe dreams

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We introduce two new bases for polynomials that lift monomial and fundamental quasisymmetric functions to the full polynomial ring. By defining a new condition on pipe dreams, called quasi-Yamanouchi, we give a positive combinatorial rule for expanding Schubert polynomials into these new bases that parallels the expansion of Schur functions into fundamental quasisymmetric functions. As a result, we obtain a refinement of the stable limits of Schubert polynomials to Stanley symmetric functions. We also give combinatorial rules for the positive structure constants of these bases that generalize the quasi-shuffle product and shuffle product, respectively. We use this to give a Littlewood–Richardson rule for expanding a product of Schubert polynomials into fundamental slide polynomials and to give formulas for products of Stanley symmetric functions in terms of Schubert structure constants.

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1. Introduction

The Schubert polynomials give explicit polynomial representatives for the Schubert classes in the cohomology ring of the complete flag variety, with the goal of facilitating computations of intersection numbers. Lascoux and Schützenberger [10] first defined Schubert polynomials indexed by permutations in terms of divided difference operators, and later Billey, Jockusch and Stanley [2] and Fomin and Stanley [6] gave direct monomial expansions. Bergeron and Billey [1] reformulated this again to give a beautiful combinatorial definition of Schubert polynomials as generating functions for RC-graphs, often called *pipe dreams*. However, even armed with these elegant formulations, the long-standing problem of giving a positive combinatorial formula for the structure constants of Schubert polynomials remains open in all but a few special cases.

In this paper, we introduce a new tool to aid in the study of Schubert polynomials. We define two new families of polynomials we call the *monomial slide polynomials* and *fundamental slide polynomials*. Both monomial and fundamental slide polynomials are combinatorially indexed by weak compositions, and both families form a basis of the polynomial ring. Moreover, the Schubert polynomials expand positively into the fundamental slide basis, which in turn expands positively into the monomial slide basis. While there are other bases that refine Schubert polynomials, most notably key polynomials [3,11], ours has two main properties that make it a compelling addition to the theory of Schubert calculus. First, our polynomials exhibit a similar stability to that of Schubert polynomials, and so they facilitate a deeper understanding of the stable limit of Schubert polynomials, which, as originally shown by Macdonald [12], are Stanley symmetric functions [14]. Second, and in sharp contrast to key polynomials, our bases themselves have *positive* structure constants, and so our Littlewood–Richardson rule for the fundamental slide expansion of a product of Schubert polynomials takes us one step closer to giving a combinatorial formula for Schubert structure constants.

To motivate our new bases, let us first recall a special case in which the Schubert problem is solved explicitly, that of the Grassmannian partial flag variety. In this case, Schubert polynomials are nothing more than Schur polynomials, which form a well-studied basis for symmetric polynomials, that is, polynomials invariant under any permutation of the variables. Schur polynomials have a beautiful combinatorial definition as the generating functions of semistandard Young tableaux, and the original Littlewood–Richardson rule gives an elegant combinatorial formula for the Schur structure constants as the number of so-called *Yamanouchi* tableaux, which are semistandard tableaux satisfying certain additional conditions. This rule has many reformulations and many beautiful proofs, yet so far none of these has been lifted to the general polynomial setting.

As an intermediate step to this lift, we consider instead the ring of quasisymmetric polynomials, that is, polynomials invariant under certain permutations of the variables. Gessel [7] defined the fundamental basis for quasisymmetric polynomials, and showed that the Schur polynomials may be written as the generating function of standard Young