Advances in Mathematics 306 (2017) 123–176

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

An $L_q(L_p)$-theory for the time fractional evolution equations with variable coefficients

Ildoo Kim [a], Kyeong-Hun Kim [b,*,1], Sungbin Lim [b]

[a] Center for Mathematical Challenges, Korea Institute for Advanced Study, 85 Hoegiro Dongdaemun-gu, Seoul, 130-722, Republic of Korea
[b] Department of Mathematics, Korea University, 1 Anam-dong, Sungbuk-gu, Seoul, 136-701, Republic of Korea

A R T I C L E   I N F O

Article history:
Received 5 December 2014
Received in revised form 26 August 2016
Accepted 31 August 2016
Available online xxxx
Communicated by Charles Fefferman

MSC:
45D05
45K05
45N05
35B65
26A33

Keywords:
Fractional diffusion-wave equation
$L_q(L_p)$-theory
L_p-theory
Caputo fractional derivative
Variable coefficients

A B S T R A C T

We introduce an $L_q(L_p)$-theory for the semilinear fractional equations of the type

$$\partial_t^\alpha u(t,x) = a^{ij}(t,x)u_{x^ix^j}(t,x) + f(t,x,u), \quad t>0,\ x\in\mathbf{R}^d. \tag{0.1}$$

Here, $\alpha \in (0,2)$, $p,q>1$, and ∂_t^α is the Caupto fractional derivative of order α. Uniqueness, existence, and $L_q(L_p)$-estimates of solutions are obtained. The leading coefficients $a^{ij}(t,x)$ are assumed to be piecewise continuous in t and uniformly continuous in x. In particular $a^{ij}(t,x)$ are allowed to be discontinuous with respect to the time variable. Our approach is based on classical tools in PDE theories such as the Marcinkiewicz interpolation theorem, the Calderon–Zygmund theorem, and perturbation arguments.

* Corresponding author.
E-mail addresses: waldoo@kias.re.kr (I. Kim), kyeonghun@korea.ac.kr (K.-H. Kim), sungbin@korea.ac.kr (S. Lim).
[1] This work was supported by Samsung Science and Technology Foundation under Project Number SSTF-BA1401-02.

http://dx.doi.org/10.1016/j.aim.2016.08.046
0001-8708/© 2016 Elsevier Inc. All rights reserved.

1. Introduction

Fractional calculus has been used in numerous areas including mathematical modeling [24,44], control engineering [6,32], electromagnetism [13,43], polymer science [3,29], hydrology [4,41], biophysics [14,22], and even finance [36,40]. See also [15,30,37,50] and references therein. The classical heat equation $\partial_t u = \Delta u$ describes the heat propagation in homogeneous mediums. The time-fractional diffusion equation $\partial_t^\alpha u = \Delta u$, $\alpha \in (0,1)$, can be used to model the anomalous diffusion exhibiting subdiffusive behavior, due to particle sticking and trapping phenomena (see [25,28]). The fractional wave equation $\partial_t u = \Delta u$, $\alpha \in (1,2)$ governs the propagation of mechanical diffusive waves in viscoelastic media (see [23]). The fractional differential equations have an another important issue in the probability theory related to non-Markovian diffusion processes with a memory (see [26,27]).

The main goal of this article is to present an $L_q(L_p)$-theory for the semilinear fractional evolution equation

$$\partial_t^\alpha u(t,x) = a^{ij}(t,x)u_{x^ix^j}(t,x) + b^i(t,x)u_{x^i}(t,x) + c(t,x)u(t,x) + f(t,x,u) \tag{1.1}$$

given for $t > 0$ and $x \in \mathbf{R}^d$. Here $\alpha \in (0,2), p, q > 1$, and ∂_t^α denotes the Caputo fractional derivative (see (2.3)). The indices i and j move from 1 to d, and the summation convention with respect to the repeated indices is assumed throughout the article. It is assumed that the leading coefficients $a^{ij}(t,x)$ are piecewise continuous in t and uniformly continuous in x, and the lower order coefficients b^i and c are only bounded measurable functions. We prove that under a mild condition on the nonlinear term $f(t,x,u)$ there exists a unique solution u to (1.1) and the $L_q(L_p)$-norms of the derivatives $D_x^\beta u$, $|\beta| \le 2$, are controlled by the $L_q(L_p)$-norm of $f(t,x,0)$.

We remark that there are a few other types of fractional derivatives such as Riemann–Liouville, Marchaud, and Grünwald–Letnikov fractional derivatives. These three fractional derivatives coincide with the Caputo fractional derivative in our solution space $\mathbb{H}^{\alpha,n}_{q,p,0}(T)$ (see [39,31] for the proof).

Here is a brief survey of closely related works. In [8,33] an $L_q(L_p)$-theory for the parabolic Volterra equations of the type

$$\frac{\partial}{\partial t}\left(c_0 u + \int_{-\infty}^{t} k_1(t-s)u(s,x)ds\right) = \Delta u + f(t,x,u), \quad t \in \mathbf{R}, x \in \mathbf{R}^d \tag{1.2}$$

is obtained under the conditions $k_1(t) \ge ct^{-\alpha}$ for small t, $c_0 \ge 0$, $\alpha \in (0,1)$, and

$$\frac{2}{\alpha q} + \frac{d}{p} < 1. \tag{1.3}$$

Note that [8,33] consider the Laplacian, i.e. $a^{ij}(t,x) = \delta^{ij}$, with the restrictions $\alpha \in (0,1)$ and (1.3). A general theory for abstract Volterra equations of the form