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Decay of eigenfunctions of elliptic PDE's, II

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A B S T R A C T

We study exponential decay rates of eigenfunctions of self-adjoint higher order elliptic operators on $\mathbb{R}^d$. We are interested in decay rates as a function of direction. We show that the possible decay rates are to a large extent determined algebraically.

Contents

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1. Introduction and previous results

Consider a real elliptic polynomial Q of degree q on $\mathbb{R}^d$. (Q elliptic means that for large $\xi \in \mathbb{R}^d, C|Q(\xi)| > |\xi|^q$ for some C.) We consider the operator $H = Q(p) + V(x)$, $p = -\mathrm{i}\nabla$, on $L^2(\mathbb{R}^d)$ with V bounded and measurable. For most of our results we assume $\lim_{|x|\to\infty} V(x) = 0$ and additional decay properties of the potential. By the assumptions on Q the operator $Q(p)$ is self-adjoint with domain the standard Sobolev space of order q which consequently is also the domain of H. The goal of the paper is to study exponential decay of L^2-eigenfunctions of H with eigenvalue $\lambda \in \mathbb{R}$ as a function of direction. It is the second in a series of two papers on exponential decay. The first one is [11].

In [2], Agmon investigated the asymptotic behavior of the Green's function (the integral kernel of the inverse of $Q(p) - \lambda$ for spectral parameter λ in the resolvent set of $Q(p)$). In certain cases he obtained rather precise asymptotics of this function. Since we are investigating the asymptotic behavior of eigenfunctions of $Q(p)+V(x)$ with $V(x)$ small at infinity, one might suspect that the asymptotic behavior of the Green's function would determine the exponential rate of fall-off of the eigenfunction. This is false in a rather spectacular way: First, the eigenvalue λ may actually be in the spectrum of $Q(p)$ where the Green's function decays (at most) like an inverse power of $|x|$ while the eigenfunction decays exponentially. And second, whether or not the eigenvalue is in the spectrum of $Q(p)$, there may be several (global or local) decay rates which occur for different potentials V of compact support. Of course at least one of these decay rates will not reflect the asymptotic behavior of the Green's function. Already in [11] we gave examples of these phenomena. For another example see Section 4. These phenomena do not occur if $Q(\xi) = |\xi|^2$, at least if for example $V = o(|x|^{-1/2})$ at infinity (see Theorems 1.3 and 3.6).

We first summarize some of the results of [11] which will be our starting point. References to previous work are given there. We define the *global decay rate* of $\phi \in L^2(\mathbb{R}^d)$ as

$$\sigma_g = \sup\{\sigma \geq 0 | \mathrm{e}^{\sigma|x|}\phi \in L^2\}. \tag{1.1}$$

It is intuitively clear that σ_g is determined by the directions of weakest exponential decay of ϕ.

In the rest of this section we assume that $(H - \lambda)\phi = 0$ with $\lambda \in \mathbb{R}$ and $\phi \in L^2(\mathbb{R}^d)$. We will mostly assume there is a splitting of V, $V = V_1 + V_2$, into bounded functions, with V_1 smooth and real-valued and V_2 measurable, with additional assumptions depending on the result.

Theorem 1.1. *Under either of the following two conditions we can conclude that $\sigma_g > 0$:*

1) $\lambda \notin \mathrm{Ran}Q := \{Q(\xi)|\xi \in \mathbb{R}^d\}$ *and $V(x) = o(1)$ at infinity.*
2) $\lambda \in \mathrm{Ran}Q$ *but λ is not a critical value of $Q(\xi)$, ξ real, and in addition*