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Pfaffian systems of A-hypergeometric equations I: Bases of twisted cohomology groups

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A B S T R A C T

We consider bases of Pfaffian systems for A-hypergeometric systems. These are given by Gröbner deformations, they also provide bases for twisted cohomology groups. For a hypergeometric system associated with a class of order polytopes, these bases have a combinatorial description. The size of the bases associated with a subclass of the order polytopes has a growth rate of polynomial order.

1. Introduction

The gamma distribution in statistics is a probability distribution on $t \in (0, +\infty)$ with two parameters $\gamma > 0$ (shape) and $x > 0$ (rate). The probability density function is written as $\exp(-xt)t^{\gamma-1}/\Phi(\gamma; x)$ where the normalizing constant Φ can be expressed in terms of the gamma function as

$$\Phi(\gamma; x) = \int_0^{+\infty} \exp(-xt)t^{\gamma-1}dt = x^{-\gamma}\Gamma(\gamma).$$

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Numerical evaluation of the Gamma function is an important problem to apply the Gamma distribution to problems in statistics. In [12,22], a new method to evaluate numerically normalizing constants for a class of unnormalized distributions was proposed. It is the holonomic gradient method (HGM). The key step of this method is to construct a Pfaffian system of differential or difference equations associated to the normalizing constant.

In a series of papers, we are going to study numerical evaluations of A-hypergeometric functions regarded as a generalization of the gamma and the beta distributions by the HGM, which leads us interesting mathematical problems. We will discuss on one of them, which is a method to construct bases of Pfaffian systems associated to A-hypergeometric functions.

Let $g(x,t) = \sum_{a\in\mathcal{A}} x_a t^a$, $t^a = t_1^{a_1}\cdots t_d^{a_d}$ be a generic sparse polynomial in $t = (t_1,\ldots,t_d)$ with the support on a finite set of points $\mathcal{A} \subset \mathbf{Z}^d$. The coefficients x_a, $a \in \mathcal{A}$ are denoted by x_i, $i = 1,\ldots,n$. The function defined by the integral

$$\Phi(x) = \int_{C_1} g(x,t)^\alpha t^\gamma dt, \quad \text{or} \quad \Phi(x) = \int_{C_2} \exp(g(x,t)) t^\gamma dt, \; dt = dt_1\cdots dt_d$$

over a cycle C_i in the t-space is called an *A-hypergeometric function* of x with parameters $\alpha \in \mathbf{C}$, $\gamma_i \in \mathbf{C}$ [1,9,11]. It is known that the A-hypergeometric function satisfies a system of linear partial differential equations in x, which is called the *A-hypergeometric system*. The A-hypergeometric system is a holonomic system, and the operators of the system generate a zero-dimensional ideal in the ring of differential operators with rational function coefficients (see, e.g., [13, Chapter 6]). A-hypergeometric systems have been studied for the past 25 years (see, e.g., [10,11,26]), and they have applications in many fields.

The function $g(x,t)^\alpha t^\gamma/\Phi(x)$ or $\exp(g(x,t))t^\gamma/\Phi(x)$ can be regarded as a probability distribution function on C_i with parameters x, α, γ satisfying certain conditions. This distribution, which we will call the *A-distribution*, is a generalization of the beta distribution or the gamma distribution. In this context, the function $\Phi(x)$ is called the normalizing constant of the A-distribution. In [12,13,22], some new statistical methods were proposed. These were the holonomic gradient method (HGM) and the holonomic gradient descent (HGD). The HGM is a method for numerically evaluating the normalizing constant, which is a function of the parameters x, for a given unnormalized probability distribution, and the HGD uses the HGM to obtain the maximum likelihood estimate. The key step for both of these methods is to construct a Pfaffian system associated with the normalizing constant. The size of the Pfaffian system determines the complexity of the HGM and the HGD (see, e.g., [19]). The HGM and HGD lead us to the following fundamental goals for applying A-hypergeometric systems to statistics.

1. Find an efficient method for constructing a Pfaffian system associated with a given A-hypergeometric system.